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Mathematical Reviews

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ALGEBRA

Riordan, John. Three-line Latin rectangles. Amer. Math. Monthly 51, 450–452 (1944). [MF 11246]

The author observes that the solution of the "problème des ménages" is equivalent to finding the number of three-line Latin rectangles whose second line is a circular permutation of the first. Using the methods of Kaplansky [Amer. Math. Monthly 46, 159–161 (1939)], he obtains a new formula for the number of three-line Latin rectangles of side n and proves that it is asymptotically equal to $(n!/e)^2$.

H. B. Mann (Columbus, Ohio).

Farnell, A. B. Limits for the characteristic roots of a matrix. Bull. Amer. Math. Soc. 50, 789–794 (1944). [MF 11297]

Let $A=(a_{rs})$ be a square matrix of order n and x a column vector of dimension n over the complex field. If x^* is the conjugate transpose of x, the totality of complex numbers $\lambda=x^*Ax$, where $x^*x=1$, constitutes the field of values of A. Since, if $Ax=\lambda x$, $x^*Ax=x^*\lambda x=\lambda x^*x=\lambda$, all characteristic numbers of A are elements of the field of values of A. Further, if $A=B+iC$, where B and C are Hermitian and $\lambda=\alpha+i\beta$ is an element of the field of values of A, α and β are elements of the fields of values of B and C, respectively. The author determines several upper limits for $|\lambda|$, $|\alpha|$ and $|\beta|$, of which the following are typical: (1) "Let $R_r=\sum_s|a_{rs}|$, $T_s=\sum_r|a_{rs}|$, $R=\max(R_r)$, $T=\max(T_s)$. Then $|\lambda|\leqq(RT)^{\frac{1}{2}}$." (2) "Let $U_r=\sum_s|a_{rs}|^2$ and $V_s=\sum_r|a_{rs}|^2$. Then $\lambda\leqq(\sum_r(U_rV_r)^{\frac{1}{2}})^{\frac{1}{2}}$." Since the proofs depend solely on the form x^*Ax, corresponding theorems are true for $|\alpha|$ and $|\beta|$.

J. Williamson (Flushing, N. Y.).

Terracini, Alejandro. Some elementary remarks concerning the reality of the roots of an algebraic equation. Math. Notae 4, 137–144 (1944). (Spanish) [MF 11481]

Tschebotaröw, N. G. The problem of resolvents and critical manifolds. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 7, 123–146 (1943). (Russian. English summary) [MF 9868]

In this paper the author considers an equation

$$f(x)=x^n+a_1x^{n-1}+\cdots+a_n,$$

whose coefficients a_i are polynomials with complex coefficients in $m(\leqq n)$ variables u_j $(j=1,\cdots,m)$, and studies the problem of determination of a resolvent $g(y)=0$ whose Galois group is isomorphic to that of $f(x)$, whose roots y_i are rationally related to the roots x_i of $f(x)$ by the equations

$$y_i=\alpha_0+\alpha_1x_i+\cdots+\alpha_{n-1}x_i^{n-1},\qquad i=1,\cdots,n,$$

and whose coefficients contain the least possible number s of parameters. He points out that this problem includes the problem of resolvents as formulated by Klein [Gesammelte Mathematische Abhandlungen, vol. 2, Berlin, 1922, pp. 255–504] and, in a more general form, by Hilbert [Math. Ann. 29, 243–250 (1927); Gesammelte Abhandlungen, vol. 2, Berlin, 1933, pp. 393–400]. The author employs a correspondence between the substitutions S of the Galois group and the (critical) manifolds U_S of the space U of the variables $u_1,\cdots,u_m$ defined as follows: if

$$S=(1,2,\cdots,\mu_1)(\mu_1+1,\cdots,\mu_2)\cdots(\mu_{k-1}+1,\cdots,\mu_k)$$

is the representation of S in cycles, then U_S is the set of those points P of U for which $x_1=x_2=\cdots=x_{\mu_1}$, $x_{\mu_1+1}=x_{\mu_1+2}=\cdots=x_{\mu_2}$, $\cdots$, $x_{\mu_{k-1}+1}=\cdots=x_{\mu_k}$. He associates with each point P of U a group (inertia group) of substitutions of the roots, induced by a variation of $(u_1,\cdots,u_m)$ through closed paths in the neighborhood of P, studies the relations between this inertia group and the critical manifolds on one hand and between inertia groups of points P in U and inertia groups of corresponding points in the space of parameters of $g(y)$ on the other, and arrives at the following result. If a_i are linear functions of u_j and if there exists a chain $U_{S_1}\supset U_{S_2}\supset\cdots\supset U_{S_q}$ of q critical manifolds of $f(x)=0$, then every rational resolvent contains not less than q parameters.

Applying this theorem to a polynomial with variable coefficients and alternating group, the author obtains the inequality $s\geqq[(n-1)/2]$. Comparing his results with those of Hilbert in the following table the author remarks that

	$n=$	5	6	7	8	9
Rational resolvent	$s\geqq[(n-1)/2]=$	2	2	3	3	4
Hilbert resolvent	$s\leqq$	1	2	3	4	4

for $n=5$ the use of irrational resolvents actually decreases the number of parameters required, that for $n=6, 7, 9$, s may be equal to $[(n-1)/2]$ and that the case of $n=8$ requires further investigation. The paper closes with some observations regarding the problem of irrational resolvents and that of determination of an upper bound for s.

A. E. Ross (St. Louis, Mo.).

Dubnov, J. et Ivanov, V. Sur l'abaissement du degré des polynômes en affineurs. C. R. (Doklady) Acad. Sci. URSS (N.S.) 41, 95–98 (1943). [MF 11054]

It is shown that, if $N\geqq 2^n-1$, the product $A_1A_2\cdots A_n$ of N matrices of degree n can be expressed as a linear combination of products of the A_i, each of which contains at most $N-1$ of the given N factors. The coefficients of this linear combination are scalars which depend on the N factors A_i.

R. Brauer (Toronto, Ont.).

Abstract Algebra

***Bourbaki, N. Éléments de mathématique. Part I. Les structures fondamentales de l'analyse. Livre II. Algèbre. Chapitre I. Structures algébriques.** Actual. Sci. Ind., no. 934. Hermann et Cie., Paris, 1942. iv+165 pp.

The contents of this chapter can be divided into two parts. The first one deals with the general concept of an

algebraic structure, while the second deals with more specific algebraic structures, such as groups, rings and fields.

An internal composition law $\top$ in a set E is a function which with some pairs x, y of elements of E associates an element $x \top y$ of E. An external composition law $\perp$ between the elements of a set Ω (called the set of operators) and the elements of a set E is a function which with some pairs $\alpha \epsilon \Omega$ and $x \epsilon E$ associates an element $\alpha \perp x$ in E. An algebraic structure is a set E with a collection of internal and external composition laws. Specific algebraic structures can be obtained by imposing axioms upon the composition laws (like commutativity, associativity) and relations between them (like distributivity). Despite this great generality such concepts as representation, substructure, quotient-structure, homomorphism, etc. are defined, and the analogues of the two isomorphism theorems of group theory are proved.

Let $\top$ be a commutative and associative internal composition law in E. An element $a \epsilon E$ is called regular if the equation $a \top x = b$ has at most one solution. The authors show that E can be imbedded in a set $\bar{E}$ and $\top$ can be extended to $\mp$ in such a fashion that $\bar{E}$ has a unit and every regular element in E has an inverse in $\bar{E}$. In a sense the extension $\bar{E}$ is defined uniquely. This theorem supplies a neat way for introducing negative integers and fractions.

The section on groups contains the usual definitions and theorems concerning groups with operators, including the Jordan-Hölder decomposition theorem. This is followed by a separate section on groups of transformations (including permutations). The elementary theory of rings and fields is taken up in the last two sections. The chapter closes with a lengthy historical note. The reader will also find a great number of exercises of various orders of difficulty following each section. *S. Eilenberg* (New York, N. Y.).

Newman, M. H. A. Axioms for algebras of Boolean type. J. London Math. Soc. 19, 28–31 (1944). [MF 11317]

The purpose of this note is to improve on one of the sets of axioms which were given in a previous paper [J. London Math. Soc. 16, 256–272 (1941); these Rev. 4, 70] for a certain type of algebra characterized as the direct union of a nonassociative Boolean ring with unit and a Boolean lattice. *O. Ore* (New Haven, Conn.).

Tihomirov, A. A new proof of a theorem concerning simple rings. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 8, 139–142 (1944). (Russian. English summary) [MF 11464]

The author proves without assuming associativity of the algebras the otherwise known theorem: if A is a simple and normal algebra and the centrum K of A is contained in the centrum of B then every two-sided ideal of the algebra $A \times B$ over K is of the form $A \times \mathfrak{b}$, where $\mathfrak{b}$ is a two-sided ideal of B. As a corollary a similar proposition is obtained for rings in which the associative law holds for every three elements at least one of which belongs to the centrum. The formulation of the theorems in the English summary is not quite accurate. *G. Y. Rainich* (Ann Arbor, Mich.).

Durfee, William H. Congruence of quadratic forms over valuation rings. Duke Math. J. 11, 687–697 (1944). [MF 11571]

E. Witt [J. Reine Angew. Math. 176, 31–44 (1936)] proved that if f, g and h are quadratic forms over a field of characteristic not 2, if g and h each have no variables in common with f, and if $f+g$ and $f+h$ are congruent and nonsingular, then g is congruent to h. The author proves that this is true over any complete valuation ring whose associated residue-class field has characteristic not 2. These results may be stated in terms of matrices. For some of the theorems the restriction "characteristic not 2" is not used but it is essential to the main theorem. *B. W. Jones.*

Vinograde, Bernard. Cleft rings. Trans. Amer. Math. Soc. 56, 494–507 (1944). [MF 11492]

If the minimum condition is satisfied by the left ideals of the (not nilpotent) ring R, then R is known to possess a nilpotent radical N. If there exists, furthermore, a subring R^* of R such that every coset of R/N contains one and only one element of R^*, then R is termed by the author a cleft ring, R^* its cleft. Wedderburn's theorem and its generalizations assure the existence of a fairly large class of algebras that are cleft rings. If e is a primitive idempotent in the cleft ring R, then eRe is a cleft ring and is, therefore, the group theoretical direct sum of its radical eNe and of a division ring eR^*e, for R^* a cleft of R. If, conversely, eRe is a cleft ring for every e in a greatest set of mutually orthogonal primitive idempotents, then R is a cleft ring. Suppose now that R and R' are rings (with unit) of endomorphisms of the Abelian group V, that each of them is the commutator ring of the other (in the ring of all endomorphisms of V), that the R-group V possesses a finite composition series, that the minimum condition is satisfied by the left ideals in R and by the right ideals in R', and that R and R' are cleft rings. If e is a primitive idempotent of R, then eV is an indecomposable R'-subgroup of V of dimension s over the division subring eR^*e of R, occurring with a multiplicity n (in a decomposition of the R'-group V); and s of the factors of the composition series of the R-group V are isomorphic to R^*e and they are of dimension n over a division subring of R'. These results are then applied to the special case $R = V$, and to algebras. *R. Baer* (Urbana, Ill.).

Jennings, S. A. A note on chain conditions in nilpotent rings and groups. Bull. Amer. Math. Soc. 50, 759–763 (1944). [MF 11292]

Let R be a group and let Ω be a set of (left) operators of R which includes the inner automorphisms of R. The group R is said to be Ω-nilpotent if there exists a strictly decreasing chain of Ω-subgroups $R = A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1} = 1$ such that A_{i+1} contains all commutator elements of an element of A_i and an element of R, and that for every λ in Ω and every a_i in A_i there exists an integer r for which λa_i lies in $a_i^r A_{i+1}$. It is shown that the maximal or minimal condition for Ω-subgroups in R implies the maximal or minimal condition, respectively, for all subgroups in R. This remains valid when R is a Φ-group with regard to a second set of operators Φ and only Φ-subgroups are admitted. As an application, rings R are considered which need not be associative. A ring R is nilpotent when there exists a strictly decreasing chain $R = A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1} = 0$ of ideals such that $RA_i \subseteq A_{i+1}$, $A_i R \supseteq A_{i+1}$. In a nilpotent ring, the maximal or minimal condition for ideals implies the corresponding condition for modules. *R. Brauer.*

Hochschild, G. On the cohomology groups of an associative algebra. Ann. of Math. (2) 46, 58–67 (1945). [MF 11788]

S. Eilenberg and S. MacLane have recently [Bull. Amer. Math. Soc. 50, 53 (1944), abstract 9] introduced cohomology groups into the theory of discrete groups. In the present

paper their definitions are carried over to the theory of algebras. If A is an algebra over the field F, and if the linear space P over F is at the same time an A-right and an A-left module (in short: an A-module), then it is possible to introduce the nth cohomology group $H^n(A, P)$ by a suitable adaptation of the definitions of Eilenberg-MacLane. A linear A to P function $f(x)$, satisfying $f(xy)=xf(y)+f(x)y$, has been termed a derivation [N. Jacobson, Trans. Amer. Math. Soc. **42**, 206–224 (1937), in particular, p. 214]; if there exists an element d in P such that $f(x)=xd-dx$, then $f(x)$ is called an inner derivation. The group of all the derivations of A in P, taken modulo the subgroup of its inner derivations, is just the first cohomology group of A in P. The linear space of all the linear mappings of A into P is made into an A-module $(A:P)$ by letting af, for a in A and f in $(A:P)$, be the mapping of x onto $af(x)$, and fa the mapping of x onto $f(ax)-f(a)x$. Then the nth cohomology group $H^n(A, P)$ is just the $(n-1)$st cohomology group $H^{n-1}(A, (A:P))$. In particular, it is possible to characterize the second cohomology groups as follows. If h is a homomorphism of some algebra B over F upon the algebra A such that the kernel K of h satisfies $K^2=0$, then K may be considered as an A-module P. The second cohomology group $H^2(A, P)$ is then essentially the same as the system of all the extensions B of P by A in the sense described above. Finally we state the two main results: $H^1(A, P)=0$ for every A-module P if, and only if, A is separable; that is, if, and only if, the algebra obtained from A by extending the field F is semi-simple for every extension of F; $H^2(A, P)=0$ for every A-module P if, and only if, there exists for every algebra B over F and for every two-sided ideal J in B, such that A and B/J are isomorphic, a subalgebra A^* of B such that every coset of B/J contains one and only one element in $A^*[\cong A]$. *R. Baer* (Urbana, Ill.).

Jacobson, N. Construction of central simple associative algebras. Ann. of Math. (2) **45**, 658–666 (1944). [MF 11374]

In the classical theory of crossed products, the notions of splitting fields and factor sets refer to a class of equivalent algebras rather than to an individual algebra. On the other hand, it is known that a central division algebra $\mathfrak{A}$ of dimension n^2 over a field Φ always contains a field P of dimension n over Φ, which may even be assumed to be separable over Φ. The author proposes to study directly the relationship between $\mathfrak{A}$ and P; more generally, he considers any algebra $\mathfrak{A}$ of dimension n^2 over Φ which contains a field P of dimension n. The algebra $\mathfrak{A}$ may then be considered as a double-module over P and therefore defines a self-representation E of P [cf. Jacobson, Amer. J. Math. **66**, 1–29 (1944); these Rev. **6**, 35]. If we take a right base $\{x_1, \cdots, x_n\}$ of $\mathfrak{A}/\mathrm{P}$, we have formulas of the type $x_ix_j=\sum x_\mu\sigma_{\mu ij}$, with $\sigma_{\mu ij}$ in P; these $\sigma_{\mu ij}$ form a "factor set." The associativity of the multiplication in $\mathfrak{A}$ gives rise to certain relations involving the $\sigma_{\mu ij}$'s and the self-representation E. Conversely, being given a separable extension P of Φ, a regular self-representation E of P (that is, a self-representation where irreducible components are contained in the self-representation yielded by the direct product of P with itself) and a factor set (σ) for E satisfying the associativity conditions found above, it is possible to construct an associative algebra $\mathfrak{A}=(\mathrm{P}, E, \sigma)$ which is called a "crossed product." If $\mathfrak{A}$ has a unit element, then it is central. Its two-sided ideals are direct sums of the sub-modules $\mathfrak{A}_i$ of $\mathfrak{A}$ considered as a double P-module. Conditions for $\mathfrak{A}$ to be simple may be formulated in terms of the $\mathfrak{A}_i$'s. The condition for $\mathfrak{A}$ to be a full matrix algebra can be given in terms of the factor set. To conclude, the author derives from his theory a generalized "Hauptgeschlechtsatz" relative to a self-representation.
C. Chevalley (Princeton, N. J.).

Jacobson, N. Relations between the composites of a field and those of a sub-field. Amer. J. Math. **66**, 636–644 (1944). [MF 11400]

In a previous paper [Jacobson, Amer. J. Math. **66**, 1–29 (1944); these Rev. **6**, 35] the author has shown that the theory of composites of a field with itself gives rise to a Galois theory for nonnormal extensions. The present paper contains a continuation of this Galois theory in the direction of the study of the relations among the Galois-hypergroups of P/Σ, Σ/Φ, P/Φ, where $\Phi\subset\Sigma\subset\mathrm{P}$, P separable over Φ. It is shown that every composite of P/Φ induces a composite of Σ/Φ, and that this mapping induces a homomorphism between the Galois hypergroups of P/Φ and of Σ/Φ. Incidentally, the author derives an analogue of Schur's lemma for self-representations in which the inverse self-representation takes the place of the contragredient representation of classical representation theory. *C. Chevalley.*

Jacobson, N. Galois theory of purely inseparable fields of exponent one. Amer. J. Math. **66**, 645–648 (1944). [MF 11401]

It is known that a Galois theory for purely inseparable extensions of rank 1 can be developed in which the derivations take the place of automorphisms [cf. Jacobson, Trans. Amer. Math. Soc. **42**, 206–224 (1937)]. This theory is presented anew, this time following the pattern of the Artin form of ordinary Galois theory. *C. Chevalley.*

Albert, A. A. Two element generation of a separable algebra. Bull. Amer. Math. Soc. **50**, 786–788 (1944). [MF 11296]

An algebra A is said to be separable if it is the direct sum of simple components A_k such that the center of every A_k is a separable field over the coefficient field F. It is shown that, if A is a separable algebra of finite order over the infinite field F, then A can be generated by a single element when A is commutative and by two elements when A is noncommutative. *O. Ore* (New Haven, Conn.).

Albert, A. A. Quasiquaternion algebras. Ann. of Math. (2) **45**, 623–638 (1944). [MF 11372]

In an earlier paper [Ann. of Math. (2) **43**, 708–723 (1942); these Rev. **4**, 186], the author introduced a new class of nonassociative central simple algebras. In the present paper, some of these algebras are studied in detail. These "quasiquaternion algebras" form generalizations of the generalized quaternion algebras. They have order four over a given field F. Let Z be an algebra of order two over F and assume that Z has a unity element e. If e and u form a basis of Z, we may assume without restriction that $u^2=\rho e+\sigma u$ (ρ, σ in F) where $\rho=0$ when the characteristic of F is not two, and $\rho=0$ or $\rho=1$ when the characteristic of F is two. Let τ be a fixed element of F with $\tau\neq 0, \rho$, and denote by $a\to a'$ the linear transformation $a=\alpha_1e+\alpha_2u\to a'=\alpha_1e+\alpha_2(\tau e-u)$. The quasiquaternion algebra A will depend on ρ, σ, τ and on a further parameter γ (with γ in F). The general element of A has the form $y+jz$ where y and . are arbitrary elements of Z and j is a fixed element of A. Multiplication in A is defined by

$$(y+jz)(a+jb)=ay+\gamma bz'+j(az+by')$$

for y, z, a, b in Z. A complete theory with remarkably simple results is given for these algebras A. The subalgebras and the automorphisms of A are determined. The conditions are given that two such algebras be isomorphic, further the condition that A be a division algebra. Finally, the condition for isotopy of two division algebras of quasiquaternion type is obtained if the characteristic of F is different from two. *R. Brauer* (Toronto, Ont.).

Uzkov, A. On a class of non-associative algebras. Rec. Math. [Mat. Sbornik] N.S. **13** (55), 71–78 (1943). (English. Russian summary) [MF 11646]

Let A be an algebra over a field K, and let $a \to R(a)$, $a \to L(a)$ be the right and left regular representations of A. For arbitrary μ_0, μ_1, ν_0, ν_1 of K, and any two elements a, b of A, denote the expression

$$\mathrm{Tr}\,\{(\mu_0 L(a)+\mu_1 R(a))(\nu_0 L(b)+\nu_1 R(b))\}$$

by $\varphi(a, b)$, and call the determinant

$$D(\mu_0\mu_1\nu_0\nu_1; A) = |\varphi(e_i, e_j)|,$$

where $e_1, e_2, \cdots, e_n$ is a basis of A over K, the "$(\mu_0\mu_1\nu_0\nu_1)$-discriminant" of A. Call the algebra A "$(\mu_0\mu_1\nu_0\nu_1)$-symmetric" if $\varphi(a, bc) = \varphi(ab, c) = \varphi(ca, b)$ for all elements a, b, c of A. In particular, a (1, 0, 1, 0)-symmetric algebra is said to be "simply symmetric." Among the results obtained by the author are the following. (1) Every associative or Lie algebra is simply symmetric. (2) A sufficient condition that a $(\mu_0\mu_1\nu_0\nu_1)$-symmetric algebra may be decomposed uniquely into a direct sum of simple subalgebras is that its $(\mu_0\mu_1\nu_0\nu_1)$-discriminant shall not equal zero. (A counterexample shows that the condition is not necessary, in contrast to the classical result.) (3) The direct product of two simply symmetric algebras is simply symmetric, and the discriminant of the product vanishes if and only if that of at least one of the direct factors vanishes. *S. A. Jennings.*

Bruck, Richard H. Some results in the theory of linear non-associative algebras. Trans. Amer. Math. Soc. **56**, 141–199 (1944). [MF 11249]

A. A. Albert has developed a theory of non-associative algebras [cf., in particular, Ann. of Math. (2) **43**, 685–723 (1942); these Rev. **4**, 186]. We have to refer to these papers for some of the concepts and notations used in the following. The author calls an algebra $\mathfrak{A}$ over a field F right-simple if it contains no proper right ideals. A necessary and sufficient condition for $\mathfrak{A}$ to be right-simple is that the right multiplications R_x generate an irreducible algebra $E(\mathfrak{R})$ of matrices; any irreducible subalgebra of the full matrix algebra of degree n appears as an algebra $E(\mathfrak{R})$ for a suitable choice of the right-simple algebra $\mathfrak{A}$ of order n. If F is a field which possesses an extension field $F(\theta)$ of degree n, then any algebra $\mathfrak{A}$ which has a right unit e has an isotope which is right-simple. Albert has shown that, under the same assumption concerning F, every algebra with a unity element possesses an isotope which is not only simple, but right-simple and left-simple. The author adds the remark that, for an arbitrary field F, an algebra $\mathfrak{A}$ with a unity element possesses a simple isotope. An algebra $\mathfrak{A}$ is said to be isotopically simple if every isotope of $\mathfrak{A}$ is simple. The terms "isotopically right-simple" and "isotopically left-simple" are defined in a similar manner. If a right-simple algebra has a right unit, then it is isotopically right-simple; if a simple algebra has a unity element, then it is isotopically simple. Part of this theorem has already been given by Albert.

Every algebra possesses an isotope $\mathfrak{B}$ with a decomposition series $\mathfrak{B} = \mathfrak{B}_0 \supset \mathfrak{B}_1 \supset \cdots \supset \mathfrak{B}_k = (0)$ in which the difference algebras $\mathfrak{C}_i = \mathfrak{B}_{i-1} - \mathfrak{B}_i$ are isotopically simple. The problem of constructing the most general algebra $\mathfrak{B}$ with given $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_k$ is trivial. However, it is an open question whether the Jordan-Hölder theorem holds. All algebras are divided into four classes. (I) All algebras with at least one right nonsingular and one left nonsingular element. As has been shown by Albert, such an algebra $\mathfrak{A}$ has an isotope $\mathfrak{A}_0$ with a unity element. Then $\mathfrak{A}$ is isotopically simple if and only if $\mathfrak{A}_0$ is simple. (II) All algebras with at least one right nonsingular element but with no left nonsingular elements. Such an algebra has an isotope with a right unit (but none with a two-sided unit) and is isotopically right-simple if and only if $\mathfrak{A}_0$ is right-simple. (III) The class similar to (II) but with the roles of "right" and "left" interchanged. (IV) All algebras with no right or left nonsingular elements. It is shown that for $n \geqq 3$ there exist isotopically simple algebras of class (I) and order n. There exist isotopically right-simple algebras of class (II), order n, for every $n \geqq 4$. The case (IV) is more difficult to treat.

Among other results, it is shown that the Lie algebra of order $n(n-1)/2$ consisting of all skew-symmetric matrices over a real field is isotopically simple. Isotopically indecomposable algebras are defined and discussed briefly. The following three sections deal with quasi-group algebras. If Q is a quasi-group, then every element p corresponds to a basis element u_p of the quasi-group algebra, and the multiplication is defined by $u_p u_q = h_{p,q} u_{pq}$, where the $h_{p,q}$ are nonzero elements of F. In particular, if the $h_{p,q}$ are all equal to 1, the author uses the term "quasi-group ring." Every quasi-group ring over a nonmodular field is a direct sum of simple algebras. Each such simple component is either isotopically simple or isotopic to a direct sum of isotopically simple algebras, each with a unity element. The quasi-group algebras can be used for the construction of algebras which are both right-simple and left-simple. After a slight modification, it yields the construction of a type of simple algebras which exists for every finite order n. The question of quasi-group extensions is studied and linked with Albert's extension theory for loops [Trans. Amer. Math. Soc. **55**, 401–419 (1944); these Rev. **6**, 42]. The following sections deal with special types of infinite algebras. In particular, a construction given in Hilbert's "Grundlagen der Geometrie" is generalized. Finally, certain special algebras are treated, specifically of orders 4 and 8, and an extension of a linear vector space by the two-group is discussed. *R. Brauer* (Toronto, Ont.).

Malcev, A. On the representations of infinite algebras. Rec. Math. [Mat. Sbornik] N.S. **13** (55), 263–286 (1943). (Russian. English summary) [MF 11653]

The author is concerned with the possibility of an isomorphic representation of an algebra by matrices. The algebras treated are infinite in the sense that they do not possess finite (linear) bases, but an important part in the discussion is played by algebras which have a finite number of generators, that is, algebras whose elements can be expressed as (noncommutative) polynomials in a finite number of them with coefficients in the basic field Δ. The method, as well as many results, parallel those in the author's study of isomorphic representations of infinite groups [Rec. Math. [Mat. Sbornik] N.S. **8** (50), 405–422 (1940); these Rev. **2**, 216]. One of the results is that for algebras with a finite number of generators an isomorphic representation is possible by matrices whose elements lie in a transcendental

For Errata of the review, please see the review of a paper by Levitzki, these Rev. 8, 435.

extension of the basic field. The middle section of the paper deals with the connection between isomorphic representation by matrices of given algebras and such representation of their direct sum and their direct and free products. The last section is devoted to periodic algebras, that is, algebras all of whose elements are roots of polynomial equations with coefficients in the basic field (this property is analogous to the property of a group all of whose elements are of finite order). Combining this property with that of having a finite number of generators it is natural to ask (in analogy to Burnside's problem in group theory) whether such an algebra is finite. The (affirmative) answer to this question forms the basis of a discussion of the structure of periodic algebras which allow an isomorphic representation by matrices. One of several results is that every isomorphic representation of a semisimple periodic algebra is equivalent to such a representation in the algebraic closure of the basic field. Another is that the radical of an isomorphically representable periodic algebra over an algebraically closed field has finite index. *G. Y. Rainich* (Ann Arbor, Mich.).

Schilling, O. F. G. Automorphisms of fields of formal power series. Bull. Amer. Math. Soc. **50**, 892–901 (1944). [MF 11560]

Let F be the field of all formal power series $\sum a_i t^i$ with coefficients in a field Ω; let G be the group of automorphisms of F that leave Ω elementwise fixed. As in MacLane's treatment of p-adic fields [Ann. of Math. (2) **40**, 423–442 (1939)], the author defines the nth pseudo-ramification group R_n to consist of all $S \epsilon G$ such that $x^S \equiv x \pmod{t^n}$ for all x in the valuation ring of F. Then $R_1 = G$, $R_1/R_2 \simeq \Omega^*$ (multiplicative) and $R_n/R_{n+1} \simeq \Omega$ (additive) for $n>1$. The author offers an alternative definition of groups G_n which he identifies with R_n; however, it seems to the reviewer that there is a confusion of notation, and that, according to the author's definition, it is G_{n-1} that should be identified with R_n. It is shown that G always has elements of finite order. [In fact there are always automorphisms of order two: $t \to t/(1+t)$ for characteristic 2, and $t \to -t$ otherwise.] The author's final result exhibits G as the group of automorphisms of a suitably defined Lie ring. *I. Kaplansky.*

NUMBER THEORY

Alaoglu, Leon and Erdös, Paul. A conjecture in elementary number theory. Bull. Amer. Math. Soc. **50**, 881–882 (1944). [MF 11558]

The paper records the checking to $n=10000$ of the following conjecture of Poulet. If $\sigma(n)$ is the sum of all divisors of n and $\varphi(n)$ is Euler's function, then for any integer n the sequence $f_0(n)=n$, $f_{2k+1}(n)=\sigma(f_{2k}(n))$, $f_{2k}(n)=\varphi(f_{2k-1}(n))$ is eventually periodic. Some results like the following are proved or indicated: for every $\epsilon>0$, $\varphi(\sigma(n))<\epsilon n$, except for a set of density 0. *B. W. Jones* (Ithaca, N. Y.).

Alaoglu, L. and Erdös, P. On highly composite and similar numbers. Trans. Amer. Math. Soc. **56**, 448–469 (1944). [MF 11490]

A number is called highly composite if it has more divisors than any smaller number; n is highly abundant if $\sigma(n)>\sigma(m)$ for all $m<n$, superabundant if $\sigma(m)/m<\sigma(n)/n$ for all $m<n$ and colossally abundant if, for some $\epsilon>0$, $\sigma(n)/n^{1+\epsilon} \geqq \sigma(m)/m^{1+\epsilon}$ for $m<n$ and $\sigma(n)/n^{1+\epsilon}>\sigma(m)/m^{1+\epsilon}$ for $m>n$. Writing $n=2^{k_2}\cdots p^{k_p}$, the authors prove that, if n is superabundant,

$$\log[(q^{k_q+1}-1)/(q^{k_q+1}-q)]>(\log q/p \log p)[1+O(\delta)],$$
$$\log[(q^{k_q+2}-1)/(q^{k_q+2}-q)]<(\log q/p \log p)[1+O(\delta)],$$

where

$$\delta=(\log\log p)^2/\log p \log q$$

if $q^x<\log p$, and

$$\delta=\log p/q^{1-\theta}\log q$$

if $q^{1-\theta}>\log p$. If n is highly composite, then

$$\log(1+1/k_q)>\log q \log 2/\log p+O(\delta),$$
$$\log[1+1/(k_q+1)]<\log q \log 2/\log p+O(\delta),$$

where $\delta=(\log\log p)^2/(\log p)^2$ if $q^x<\log p$, and $\delta=1/q^{1-\theta}\log p$ if $q^{1-\theta}>\log p$. These formulas determine k_q with an error of at most 1 and in most cases uniquely.

Several related results are proved and some questions asked including the following: if p and q are different primes, is it true that p^x and q^x are both rational only if x is an integer? A table of highly abundant numbers less than 10^4 and superabundant and colossally abundant numbers less than 10^{18} is included. *B. W. Jones* (Ithaca, N. Y.).

Segre, B. A complete parametric solution of certain homogeneous Diophantine equations, of degree n in $n+1$ variables. J. London Math. Soc. **19**, 46–55 (1944). [MF 11321]

Consider the function

$$\Phi(x)=\prod_{i=1}^{n+1}(x_0+x_1t_i+\cdots+x_nt_i^n)$$

of degree $n+1$ in the variables $x_0, \cdots, x_n$, where the t_i are the roots of any polynomial equation of degree $n+1$ in 1 variable with rational coefficients which is irreducible in the rational number field. The Diophantine equations studied are

$$\sum_{i=0}^{n} b_i\partial\Phi(x)/\partial x_i=0,$$

the b_i being arbitrary rational numbers. A rational parametric solution is given for any such equation. The result is extended to the case where the t_i are roots of a reducible equation, although it is still assumed that the discriminant of the equation is not zero. An application of the result to algebraic geometry is also given. *I. Niven.*

Vandiver, H. S. On trinomial congruences and Fermat's last theorem. Proc. Nat. Acad. Sci. U. S. A. **30**, 368–370 (1944). [MF 11369]

The author states and generalizes certain results of König and of Dickson on the number of distinct roots of an equation in a finite field. Thus he obtains a formula for the number of solutions in the finite field $F(p^n)$ of the equation $f(x_1, \cdots, x_n)=0$, f being a polynomial with coefficients in F. He then applies these results to obtain criteria for Fermat's last theorem. It is possible to do this since such criteria have been found in terms of the solubility of certain congruences, that is, equations in a finite field. A formula is also obtained for the number of distinct solutions (u^m, v^m) of the congruence $au^m+bv^m+1\equiv 0 \pmod p$, where m is odd and the prime p is prime to $abuv$. *H. W. Brinkmann.*

Vandiver, H. S. Some theorems in finite field theory with applications to Fermat's last theorem. Proc. Nat. Acad. Sci. U. S. A. **30**, 362–367 (1944). [MF 11368]

The result on the trinomial congruence referred to in the preceding review is here applied to the first case of Fermat's

equation $x^l+y^l+z^l=0$, p being a prime of the form $p=cl+1$. Using a method due to H. H. Mitchell it is also proved that Fermat's equation is impossible (in the first case) if c is even and not divisible by 3, $l>c$, and $p>3^{\varphi(c)}$.

H. W. Brinkmann (Swarthmore, Pa.).

Simons, William H. Congruences involving the partition function $p(n)$. Bull. Amer. Math. Soc. **50**, 883–892 (1944). [MF 11559]

The congruence properties are for the moduli 13 and 17. They are essentially recursive congruences and they involve either $\sigma_k(m)$ or Ramanujan's $\tau(n)$. The method of proof is similar to that used by Ramanujan for the moduli 5, 7, and 11 and it is based on Ramanujan's identities connecting the functions of the type $\sum_{n=1}^{\infty} n^r\sigma_{s-r}(n)x^n$.

H. S. Zuckerman (Seattle, Wash.).

Schoenfeld, Lowell. A transformation formula in the theory of partitions. Duke Math. J. **11**, 873–887 (1944). [MF 11589]

The function

$$f_\kappa(x)=\prod_{n=1}^{\infty}(1-x^{n^\kappa})^{-1}$$

is the generating function of the number $p_\kappa(n)$ of partitions of n into κth powers. E. M. Wright [Acta Math. **63**, 143–191 (1934)] has obtained the transformation formula for this function, exhibiting its behavior near the rational points on the unit circle, and used it to determine the asymptotic behavior of $p_\kappa(n)$. Here a new and simpler proof of the transformation formula is given. The proof is a generalization of that used by Rademacher [J. Reine Angew. Math. **167**, 312–336 (1932)] for the case $\kappa=1$.

H. S. Zuckerman (Seattle, Wash.).

Ramanathan, K. G. Some applications of Ramanujan's trigonometrical sum $C_m(n)$. Proc. Indian Acad. Sci., Sect. A. **20**, 62–69 (1944). [MF 11269]

Ramanujan's sum is the sum of the nth powers of the primitive mth roots of unity:

$$C_m(n)=\mu(m/d)\varphi(m)/\varphi(m/d),$$

where d is the greatest common divisor (m, n) of m and n. The paper uses $C_m(n)$ together with well-known inversion formulas to give simple proofs of certain results of von Sterneck and Vaidyanathaswamy. The former's results are related to the theory of partitions modulo m. The following result is typical. The number of ways of selecting k distinct parts from $0, 1, \cdots, m-1$ so that their sum is congruent to n modulo m is

$$\frac{(-1)^k}{m}\sum_{\delta\mid(k,m)}(-1)^{k/\delta}\binom{m/\delta}{k/\delta}C_\delta(n).$$

Let $d_1, \cdots, d_r$ be the distinct divisors of m. Define $K(d_r)$ as the class of all numbers $n\leqq m$ for which $(n, m)=d_r$. Following Vaidyanathaswamy, define the product $K(d_i)K(d_j)$ as the class obtained by adding every member of $K(d_i)$ to every member of $K(d_j)$ and reducing the sum modulo m. It may be shown that in the resulting product set the numbers fall into complete classes $K(d_k)$ $(k=1, 2, \cdots)$. The frequency with which each class $K(d_k)$ occurs being denoted by $\gamma_{ij}^{(k)}$, we may write

$$K(d_i)K(d_j)=\sum_{k=1}^{r}\gamma_{ij}^{(k)}K(d_k).$$

The author shows that

$$\gamma_{ij}^{(k)}=m^{-1}\sum_{\delta\mid m}C_{m/d_i}(\delta)C_{m/d_j}(\delta)C_{m/\delta}(d_k).$$

This result, which occurs in cyclotomy, is generalized to the case of the product of any number of K's. As a corollary, if m is odd, every number is the sum modulo m of two parts both prime to m.

D. H. Lehmer (Berkeley, Calif.).

Haviland, E. K. An analogue of Euler's φ-function. Duke Math. J. **11**, 869–872 (1944). [MF 11588]

For a positive integer k let $\rho(k)$ denote the number of those arithmetic progressions $mk+l$ $(l=0, 1, \cdots, m-1)$ which contain infinitely many square-free numbers. It is seen that

$$\rho(k)/k=\prod_{p^2\mid k}(1-p^{-2}).$$

The author proves that $\rho(k)/k$ is a Bohr almost periodic function with an absolutely and uniformly convergent Fourier series.

M. Kac (Ithaca, N. Y.).

Wintner, Aurel. The Lebesgue constants of Möbius' inversion. Duke Math. J. **11**, 853–867 (1944). [MF 11587]

Let f and f' be connected by the relation

$$f(n)=\sum_{d\mid n}f'(d)$$

and the corresponding Möbius inverse. The author considers the problem of the equality of

$$\lim_{n\to\infty}(f(1)+f(2)+\cdots+f(n))/n$$

and $\sum_{n=1}^{\infty}f'(n)/n$. He determines the exact order of the "Lebesgue constants" of the linear transformations connecting these two expressions, thus showing that the existence of one does not necessarily imply the existence of the other. He also obtains a Tauberian restriction under which the existence of one does imply that of the other. The special case $f'(n)=\mu(n)$ shows the relation of the problem to the prime number theorem. The limits of summation on page 859 are incorrect but the final result is correct. Formula (37) should read $f'(p^k)=f(p)$. This necessitates a number of minor changes in the following work but does not affect the validity of the corollary.

H. S. Zuckerman.

Roussel, André. Sur une application d'un principe d'extremum à certaines questions d'arithmétique. C. R. Acad. Sci. Paris **217**, 496–497 (1943). [MF 11667]

It is shown that a necessary and sufficient condition that p be a prime is

$$\int_2^{p-1}(\cos\pi x\cos(\pi p/x))^{2m}dx<(1-1/(2p^6))^m,$$

where m is any value not less than $4p^6\log(8p^7)$. A similar condition is given that a set of integers be relatively prime in pairs.

I. Niven (Lafayette, Ind.).

Wang, Fu Traing. A formula on Riemann zeta function. Ann. of Math. (2) **46**, 88–92 (1945). [MF 11791]

Put

$$\Phi(z)=\prod_{\nu=1}^{\infty}(1-z^2/\rho_\nu^2),$$

where $\rho_\nu=\gamma_\nu+i\beta_\nu$ are the nontrivial zeros of $\zeta(s)$; let $N(T)$ be the number of zeros in the rectangle $0\leqq\sigma\leqq1$, $1\leqq t\leqq T$, and let $\lambda_\nu=|\rho_\nu|$, $\alpha_\nu=|\beta_\nu|/\lambda_\nu$; then the author proves

$$\int_0^T x^{-5/2}\log|\Phi(x)|dx$$
$$=(4\pi/3)\sum_{\nu=1}^{\infty}\lambda_\nu^{-1}(2\alpha_\nu-1)(1+2\nu)^{\frac12}+O(T^{-\frac12}\log T).$$

A more general formula is stated at the end of the page.

L. Carlitz (Durham, N. C.).

Tchudakoff, N. On certain sums occurring in the analytic theory of numbers. C. R. (Doklady) Acad. Sci. URSS (N.S.) 42, 326–330 (1944). [MF 11641]

Put

$$S_n = \sum_{m \leq n} \chi_0(m)\Lambda(m)\rho^m,$$

where $\chi_0(m)$ is the principal character (mod q) and ρ is a primitive qth root of unity. The author develops a new method for the evaluation of S_n and similar sums connected with prime numbers. First S_n is expressed in terms of integrals involving the logarithmic derivative of $L(S, \chi)$. Using results of A. Page [Proc. London Math. Soc. (2) 39, 116–141 (1935)] and C. L. Siegel [Acta Arith. 1, 83–86 (1935)], as well as some classical results on L'/L, an application of Cauchy's theorem finally leads to

$$S_n = n\mu(q)/\varphi(q) + O\{n \exp(-c_1(\log n)^{\frac{1}{2}})\}.$$

Applications to various prime number problems are indicated. *L. Carlitz* (Durham, N. C.).

Mahler, K. On lattice points in the domain $|xy| \leqq 1$, $|x+y| \leqq \sqrt{5}$ and applications to asymptotic formulae in lattice point theory. I. Proc. Cambridge Philos. Soc. 40, 107–116 (1944). [MF 10781]

Mahler, K. On lattice points in the domain $|xy| \leqq 1$, $|x+y| \leqq \sqrt{5}$ and applications to asymptotic formulae in lattice point theory. II. Proc. Cambridge Philos. Soc. 40, 116–120 (1944). [MF 10782]

Mahler, K. On lattice points in an infinite star domain. J. London Math. Soc. 18, 233–238 (1943). [MF 10878]

The author defined [J. London Math. Soc. 17, 130–133 (1942); these Rev. 4, 212] a finite star domain K in the (x, y)-plane as a closed set of points symmetric in $O = (0, 0)$, containing the origin O in its interior and bounded by a Jordan curve which does not intersect more than once any radius vector from O. A domain K is said to be an infinite star domain if its points in every circle of radius r about O form a finite star domain K_r. The lattice Λ of points $(x, y) = (\alpha h + \beta k, \gamma h + \delta k)$, where $h, k = 0, \pm 1, \cdots$, of determinant $d(\Lambda) = |\alpha\delta - \beta\gamma|$ is called K-admissible if the origin O is the only point of Λ interior to K. The lower bound of $d(\Lambda)$ for all K-admissible Λ is denoted by $\Delta(K)$. A K-admissible Λ for which $d(\Lambda) = \Delta(\Lambda)$ is called a critical lattice.

In the first paper of the above series, the author proves (an extension of a result of Hurwitz) that $\Delta(K_1) = \sqrt{5}$ if K_1 is the finite star domain $|xy| \leqq 1$, $|x+y| \leqq \sqrt{5}$, determines all the critical lattices for this domain, and shows that, if a star domain H is a proper part of K_1, then $\Delta(H) < \sqrt{5} = \Delta(K)$. In the second paper he considers Mordell's domain G: $|x|^\alpha + |y|^\alpha \leqq 1$, and, employing the above results, derives the simple asymptotic formula $\Delta(G) \sim 2^{-2/\alpha}\sqrt{5}$ for small α. As another application of these results, he obtains an asymptotic expression for the upper bound of the minima of positive quartic forms f of discriminant unity, when the absolute invariant J tends to infinity.

In the third paper the author considers infinite plane star domains K and shows that such domains possess admissible lattices if and only if $\lim_{r\to\infty}\Delta(K_r)$ is finite, in which case there exists a critical lattice Λ and $d(\Lambda) = \Delta(K) = \lim_{r\to\infty}\Delta(K_r)$. Employing a deep result of Markoff on the minima of indefinite binary quadratic forms, he shows that, unlike the case of finite star domains, there exist infinite star domains K^* whose critical lattices have no points on the boundary of K^*. A note at the end of this paper indicates that the author has extended his results to spaces of more than two dimensions. *A. E. Ross* (St. Louis, Mo.).

Webber, G. Cuthbert. Transcendence of certain continued fractions. Bull. Amer. Math. Soc. 50, 736–740 (1944). [MF 11287]

The following results on special continued fractions of Hurwitz's type [A. Hurwitz, Mathematische Werke, vol. 2, Basel, 1933, pp. 276–302] are proved.

(1) Let $a_1, a_2, \cdots, a_{k-1}$ be ∓ 1; let all numbers $f_1, f_2, \cdots$ be $+1$ or all be -1; let $b_1, b_2, \cdots, b_{k-1}$ be arbitrary positive integers; and let $g_n = g_0 + dn$, where $g_0 \geqq 0$ and $d > 0$ are integers. Then

$$g_0 + \frac{a_1|}{|b_1} + \cdots + \frac{a_{k-1}|}{|b_{k-1}} + \frac{f_1|}{|g_1} + \frac{a_1|}{|b_1} + \cdots + \frac{a_{k-1}|}{|b_{k-1}} + \frac{f_2|}{|g_2} + \cdots$$

is a transcendental number. (2) If a, c, d are positive integers, and either c or d is even, and if further $b = c^2a + c^2d/2$, $e = c^2d$, then

$$\frac{1|}{|a} + \frac{1|}{|b} + \frac{1|}{|a+d} + \frac{1|}{|b+e} + \frac{1|}{|a+2d} + \frac{1|}{|b+2e} + \cdots$$

is transcendental. (3) If u^2 is a positive integer, then

$$\frac{1|}{|1} + \frac{1|}{|3u^2} + \frac{1|}{|5} + \frac{1|}{|7u^2} + \frac{1|}{|9} + \frac{1|}{|11u^2} + \frac{1|}{|13} + \cdots$$

and

$$\frac{1|}{|1} + \frac{1|}{|3u^2-2} + \frac{1|}{|1} + \frac{1|}{|3} + \frac{1|}{|1} + \frac{1|}{|7u^2-2} + \frac{1|}{|1} + \frac{1|}{|7}$$
$$+ \frac{1|}{|1} + \frac{1|}{|11u^2-2} + \frac{1|}{|1} + \frac{1|}{|11} + \cdots$$

are transcendental. For the proof of these theorems, the author transforms the classical continued fractions of $(e^2 - e^{-2})/(e^2 + e^{-2})$ and $\mathfrak{F}_{\lambda-1}(2)/\mathfrak{F}_\lambda(2)$ and applies the results of Lindemann and C. L. Siegel [Abh. Preuss. Akad. Wiss., Phys.-Math. Kl. 1929, no. 1] on the transcendency of these functions. *K. Mahler* (Manchester).

Mian, A. M. and Chowla, S. The differential equations satisfied by certain functions. J. Indian Math. Soc. (N.S.) 8, 27–28 (1944). [MF 11275]

The authors prove that $\sum_1^\infty p(n)x^n$ satisfies an algebraic differential equation. *P. Erdös* (Ann Arbor, Mich.).

ANALYSIS

Theory of Sets, Theory of Functions of Real Variables

Pondiczery, E. S. Power problems in abstract spaces. Duke Math. J. 11, 835–837 (1944). [MF 11584]

Let X be any Hausdorff set and $\aleph$ any cardinal; $X^\aleph$ denotes the Cartesian product of X by itself to $\aleph$ factors. The topology of $X^\aleph$ is defined in the customary way in terms of the open sets of X. If X_α is the least cardinal of a dense subset of X, α is said to be the density index of X. The following theorem is proved. The density index of $X^\aleph$ is the greater of the following two ordinals: (1) the density index of X; (2) the least solution λ of $2^{\aleph_\lambda} \geqq \aleph$. *H. Blumberg.*

Reed, Sidney G., Jr. On some zero-dimensional sets. Rep. Math. Colloquium (2) 5–6, 36–39 (1944). [MF 11404]

This paper concerns the set T of points of Euclidean 3-space with just one rational coordinate. Now T is known to be 0-dimensional; hence each point of it is enclosable in an infinitesimal neighborhood with boundaries not meeting T. However, if a surface is of the form $y=f(x,z)$ or $x=f(y,z)$, with f continuous, it necessarily meets S. The purpose of the paper is to define a surface not meeting S.

H. Blumberg (Columbus, Ohio).

Cotlar, Mischa and Levi, Beppo. Considerations concerning a proposition of W. H. Young. Math. Notae 4, 145–155 (1944). (Spanish) [MF 11482]

The proposition [Hobson, Theory of Functions of a Real Variable, vol. 1, Cambridge, 1927, p. 183, §136] states that, if $\{G_n\}$ is a sequence of sets each of which is contained in the same closed set of finite measure, and if the inner measure of each G_n is greater than a fixed positive number C, then there is a set E of inner measure at least C, such that each point of E belongs to an infinite number of the G_n. Young's original statement [Proc. London Math. Soc. (2) 2, 16–51 (1905), in particular, p. 26] was that E itself belongs to an infinite number of the G_n. The authors show that Young's form of the theorem must be incorrect, by showing that it would imply the uniform convergence, in a set of measure 2π, of a sequence $\{\cos n_k x\}$ (n_k integral, $n_k\to\infty$); but no sequence of cosines has this property.

R. P. Boas, Jr. (Cambridge, Mass.).

Morse, Anthony P. The role of internal families in measure theory. Bull. Amer. Math. Soc. 50, 723–728 (1944). [MF 11284]

If R is a family of sets, then R_σ is the family of all sets of the form $\sum_{\beta\epsilon H}\beta$, where H is a countable nonvacuous subfamily of R; R_δ is the family of all sets of the form $\prod_{\beta\epsilon H}\beta$, where H is a countable nonvacuous subfamily of R; R_γ is the family of all sets of the form $(\sum_{\alpha\epsilon H}\alpha)-\beta$, where $\beta\epsilon R$; R^γ is the smallest countably additive family which contains R and is complemental with respect to $\sum_{\alpha\epsilon R}\alpha$; R^δ is the smallest countably multiplicative, countably additive family which contains R. Now R is internal if and only if R_δ is finitely additive and $R_\gamma\subset R^\delta$. The author establishes several theorems of the following type. If R is an internal family of ϕ measurable sets, such that ϕ measures $\sum_{\alpha\epsilon R}\alpha$, $B\epsilon R^\gamma$, $\phi(B)<\infty$, $\epsilon>0$, then B contains such a member C of R_δ that $\phi(B-C)<\epsilon$.

C. C. Torrance (Washington, D. C.).

Hadwiger, H. Ein Ueberdeckungssatz für den Euklidischen Raum. Portugaliae Math. 4, 140–144 (1944). [MF 11156]

A subset S of Euclidean n-space E_n is said to be a Δ-set if all possible distances are realized in it (that is, for every positive Δ, there exists a pair of points of S at distance Δ). The purpose of the paper is to prove the theorem: of $n+1$ closed sets covering E_n at least one is a Δ-set. The proof depends on a lemma concerning a similar covering of an n-dimensional sphere. The argument is strongly metric in character.

H. Blumberg (Columbus, Ohio).

Rothberger, Fritz. On families of real functions with a denumerable base. Ann. of Math. (2) 45, 397–406 (1944). [MF 10917]

A family F of (real) functions, defined on a range $X=\{x\}$, is said to have a denumerable base if there exists a sequence Σ of functions $f_n(x)$, $n=1, 2, \cdots$ (not necessarily belonging to F), such that each function of F is the limit of a subsequence of Σ. Sierpiński posed the following problem: if a and b are two cardinals, does every family F of cardinal a of (real) functions on a range X of cardinal b necessarily have a denumerable base? The author gives an affirmative solution to this problem for the case $a=b=\aleph_1$. (Under the continuum hypothesis, the general problem of Sierpiński is thus disposed of.) The proof depends on several lemmas concerning sets of positive integers, one of which is related to Hausdorff's first "Einschaltungssatz." The paper includes also the affirmative solution, under the continuum hypothesis, of the following problem of Ulam: if F is a family of point sets (in a Euclidean space) of cardinal of the continuum, does there exist a denumerable family D of sets such that $F\subset B(D)$? Here $B(D)$ is the smallest family containing D and closed with respect to the Hausdorff operations σ and δ.

H. Blumberg (Columbus, Ohio).

Valentine, F. A. Contractions in non-Euclidean spaces. Bull. Amer. Math. Soc. 50, 710–713 (1944). [MF 11280]

The existence of "an extension of the range of definition of a function $f(x)$ defined on a set S of a metric space M to a metric space M'," which preserves the contraction dist. $(f(x_1), f(x_2))\leqq$ dist. (x_1, x_2), is proved for M, M' n-dimensional hyperbolic. The author had previously shown [Bull. Amer. Math. Soc. 49, 100–108 (1943); these Rev. 4, 269] that a necessary and sufficient condition for an extension of this kind is the validity of a property E which insures a nonvacuous product for a certain set of spheres in M' whenever the same is true for a corresponding set of spheres in M. Using Helly's theorem on convex bodies and the well-known intersection theorem of Knaster-Kuratowski-Mazurkiewicz, it is easily seen that spaces including both the hemispherical and hyperbolic have this relative property E.

L. M. Blumenthal (Columbia, Mo.).

Amerio, Luigi. Sulla definizione di integrale di Lebesgue. Boll. Un. Mat. Ital. (2) 4, 5 pp. (1942) = Ist. Naz. Appl. Calcolo (2) no. 143. [MF 11512]

The volume no. was quoted from a reprint. The paper actually appeared in vol. 5, pp. 27–31

Scharf, Henryk. Über links- und rechtsseitige Stieltjesintegrale und deren Anwendungen. Portugaliae Math. 4, 73–118 (1944). [MF 11267]

This paper develops the properties of the Cauchy-Stieltjes integrals, defined by

$$\overset{(-)}{\int_a^b} f(x)dg(x)=\lim \sum_{i=0}^{n-1} f(x_i)[g(x_{i+1})-g(x_i)]$$

and

$$\overset{(+)}{\int_a^b} f(x)dg(x)=\lim \sum_{i=0}^{n-1} f(x_{i+1})[g(x_{i+1})-g(x_i)],$$

where $(a=x_0, x_1, \cdots, x_n=b)$ is a subdivision of (a, b) and limit is taken in the norm sense, that is, as max $(x_{i+1}-x_i)\to 0$ [see also G. B. Price, Bull. Amer. Math. Soc. 49, 625–630 (1943); these Rev. 5, 2]. It is assumed throughout that $g(x)$ is of bounded variation. An $\overset{(+)}{\int}$ can be obtained from an $\underset{(-)}{\int}$ by reflexion ($x'=-x$), also by integration by parts since

$$\overset{(-)}{\int} fdg=fg|_a^b-\overset{(+)}{\int} gdf.$$

The existence theorem for $\underset{(-)}{\int} fdg$ states that it is necessary

and sufficient that: (1) if $g(x-0)\neq g(x)$, then $f(x-0)$ exists; (2) the measure of the points at which $g(x)$ is discontinuous on the left relative to the total variation of

$$g_1(x)=g(x)-\sum_{\xi\leq x}[g(\xi)-g(\xi-0)]$$

is zero. If $\overset{(-)}{\int} fdg$ exists then it differs from the Lebesgue-Stieltjes integral in terms involving the common discontinuities of f and g. The Cauchy integrals are used to unify in an elegant manner continuous and discontinuous cases in problems occurring in life insurance. The possibility of using Riemann-Stieltjes integrals or some modification of them in such problems has been previously discussed [e.g., Steffensen, J. Inst. Actuaries 63, 443–483 (1932); de Finetti and Jacob, Giorn. Ist. Ital. Attuari 6, 303–319 (1935)].

T. H. Hildebrandt (Ann Arbor, Mich.).

Huskey, Harry D. Fréchet polyhedra. Duke Math. J. 11, 417–425 (1944). [MF 11077]

The author begins by defining Fréchet surfaces and then giving three definitions of Fréchet polyhedra. Each of these definitions leads to a corresponding definition of Lebesgue area and a fourth definition is also given, this being of a slightly different sort. The author then proves that the different definitions of area are equivalent, from which he deduces that the Lebesgue area is independent of the representation of the surface. This last result is not new but the intervening discussion of the various definitions of polyhedra (all found in the literature) is of considerable interest.

C. B. Morrey, Jr. (Aberdeen, Md.).

Young, L. C. An expression connected with the area of a surface $z=F(x, y)$. Duke Math. J. 11, 43–57 (1944). [MF 10146]

This paper supplements the chapter on the theory of area in Saks, "Theory of the Integral" [Warsaw, 1937]. Let $z=F(x, y)$ be a continuous surface defined over an interval I_0 of the (x, y)-plane and let S be the area of this surface. A well-known theorem of Tonelli states the necessary and sufficient condition that S be given by the ordinary expression

$$\int\int_{I_0}\{F_x^2+F_y^2+1\}^{\frac{1}{2}}dxdy$$

for area. For any continuous $F(x, y)$, the author considers instead

$$S_{\alpha\beta}^*=\int\int_{I_0}\left\{\left(\frac{F(x+\alpha, y)-F(x, y)}{\alpha}\right)^2+\left(\frac{F(x, y+\beta)-F(x, y)}{\beta}\right)^2+1\right\}^{\frac{1}{2}}dxdy,$$

in which difference quotients replace derivatives in the ordinary area integral. The author obtains

$$S\leq\liminf_{\alpha,\beta\to 0} S_{\alpha\beta}^*$$

and estimates of $\limsup_{\alpha,\beta\to 0} S_{\alpha,\beta}^*$ from which one may often decide whether

$$S=\lim_{\alpha,\beta\to 0} S_{\alpha,\beta}^*.$$

[In Saks' book, this equality is erroneously stated as being always valid.] Examples of surfaces are constructed to show that it may actually happen that

$$S<\liminf_{\alpha,\beta\to 0} S_{\alpha,\beta}^*.$$

These examples are built out of surfaces of the type $z=g(x+y)$, where $g(x)$ is a Cantor middle third function.

M. Shiffman (New York, N. Y.).

Radó, Tibor. Functions of rectangles. Duke Math. J. 11, 487–496 (1944). [MF 11085]

Consider a surface S: $Z=f(x, y)$ with (x, y) in the unit square Q and the integrals

$$G_1(R)=\int_a^b|f(x, d)-f(x, c)|dx,$$

$$G_2(R)=\int_c^d|f(b, y)-f(a, y)|dy,$$

$$H_i(R)=\int_R G_i \qquad (i=1, 2),$$

where R is an arbitrary oriented rectangle in Q, and

$$I(\alpha, \beta)=\int\int_Q[1+(f(x+\alpha, y)-f(x, y))^2/\alpha^2+(f(x, y+\beta)-f(x, y))^2/\beta^2]^{\frac{1}{2}}dxdy.$$

It was recently shown by L. C. Young [cf. the preceding review] that if the Lebesgue area $A(Q)$ of S is finite, then $\lim_{\alpha,\beta\to 0}I(\alpha, \beta)=A(Q)$ if and only if Q can be split into two Borel sets B_1, B_2 such that $H_i(E)$ is absolutely continuous on $B_i (i=1, 2)$, where $H_i(E)$ is the completely additive extension of $H_i(R)$ to Borel sets. In establishing this result Young makes extensive use of the theory of functions of rectangles and of completely additive extensions of functions. The author proceeds farther in this direction and is thereby able to establish one of the essential parts of Young's argument by means of an important result on rectangle functions. His lemma is concerned with three nonnegative additive and continuous rectangle functions $\varphi_j(R)$ $(j=1, 2, 3)$ of oriented rectangles R in Q and their extensions $\varphi_j(E)$ to the class of all Borel sets in Q. It is then shown that the relation $\varphi_1(E)+\varphi_2(E)+\varphi_3(E)=\beta(E)$ will hold if and only if there is a decomposition of E into three disjoint Borel sets such that $E=E_1+E_2+E_3$, $\varphi_j(E_k)=0$ $(j\neq k)$, where $\beta(E)$ is the completely additive extension of

$$\beta(R)=\int_R\psi, \quad \psi(R)=[\varphi_1(R)^2+\varphi_2(R)^2+\varphi_3(R)^2]^{\frac{1}{2}},$$

and the integral is in the sense of Burkill. Several of Young's rectangle functions are representable as integrals, the integrability of whose integrands is discussed in the last section of this paper.

H. H. Goldstine.

Radó, Tibor. Some remarks on the problem of Geöcze. Duke Math. J. 11, 497–506 (1944). [MF 11086]

Let S be a continuous surface $Z=f(x, y)$ of Lebesgue area $A(S)$, let P_n be a sequence of polyhedra inscribed in S and converging in Fréchet's sense to S and finally let $A^*(S)$ $=$ g. l. b. $\liminf E(P_n)$ where $E(P_n)$ is the ordinary area of P_n. By means of some results of L. C. Young, Huskey [Duke Math. J. 11, 333–339 (1944); these Rev. 6, 45] obtained results on the problem of Geöcze that are considerably more general than those previously obtained. This paper utilizes Huskey's methods to develop the relation-

ships between Young's integral and the area of the inscribed polyhedra P_n. It is shown that $A(R) \leqq A^*(R) \leqq 2^{\frac{1}{2}} A(R)$ for R an oriented rectangle and that $A(R)$, $A^*(R)$ have $(1+f_x^2+f_y^2)^{\frac{1}{2}}$ as their derivative. If $f(x, y)$ is absolutely continuous in Young's sense on Q then $A^*(Q) = A(Q)$ and there is a sequence of inscribed polyhedra whose areas converge to $A(S)$. The author shows that the sequence $\{x_n, y_n\}$ of subdivisions of Q, defining the polyhedra, is completely regular. *H. H. Goldstine.*

Santaló, L. A. Bounds for the length of a curve or for the number of points necessary for an approximate covering of a domain. Anais Acad. Brasil. Ci. 16, 111–121 (1944). (Spanish) [MF 11004]

(1) Let M be a bounded point set in Euclidean n-space E_n, and let μ be the measure of its closure. The distance of any point of M from the rectifiable curve $C(\epsilon)$ shall be not larger than a given $\epsilon > 0$. Then $L(\epsilon)$ shall denote the lower limit of the lengths of all the curves $C(\epsilon)$. In the case $n=2$, Kershner proved that $\lim_{\epsilon\to 0} 2\epsilon L(\epsilon) = \mu$ [Amer. J. Math. 62, 324–345 (1940); these Rev. 1, 242]. The author proves for general n that

$$\liminf_{\epsilon\to 0} L(\epsilon) V_{n-1}(\epsilon) \geqq \mu,$$

$$\limsup_{\epsilon\to 0} L(\epsilon) V_{n-1}(\epsilon) \leqq (n-1)^{(n-1)/2} \cdot \mu,$$

where $V_n(\epsilon)$ is the volume of the n-dimensional sphere of radius ϵ.

(2) Let D be a bounded domain in E_n whose boundary shall be a two times differentiable hypersurface; E_n shall be divided into hypercubes whose edges are parallel and have the same length η. The author gives an upper bound for the mean value of the number of hypercubes that intersect D if D is subjected to all the Euclidean motions, and observes that the minimum number of hypercubes of edge η that are necessary to cover D is not larger than this mean value. *P. Scherk* (Saskatoon, Sask.).

Theory of Functions of Complex Variables

✶Artin, Emil. On the theory of complex functions. Notre Dame Mathematical Lectures, no. 4, pp. 55–70. University of Notre Dame, Notre Dame, Ind., 1944.

Author's introduction: "The following pages are intended to exhibit some of the advantages obtained by a more extensive use of topological methods and notions in courses on complex variables. These methods simplify the proofs and are more flexible in their application." About three quarters of the paper is devoted to derivations of the Cauchy integral theorem (after assuming it for a triangle) and the Cauchy integral formula, proved for rectifiable closed curves, not necessarily simple (the curve may make several turns about a given point). A topological aid is the winding number of a curve C at a point not on C, that is, the number of turns which C makes about the point. Derivations of well-known results on analytic functions given in the later part of the paper, while efficient, make no unusual use of topological methods. *A. B. Brown* (Flushing, N. Y.).

Loomis, Lynn H. An elementary proof of the strong form of the Cauchy theorem. Bull. Amer. Math. Soc. 50, 831–833 (1944). [MF 11551]

A short proof is presented that for a given rectifiable Jordan curve J there is a sequence of Jordan polygons lying inside of J, converging to J, and with uniformly bounded lengths. It is then shown that this leads to a short proof of the strong form of Cauchy's theorem. *A. C. Schaeffer.*

Robinson, Raphael M. Hadamard's three circles theorem. Bull. Amer. Math. Soc. 50, 795–802 (1944). [MF 11485]

Three problems which stem from the Hadamard three circles theorem are discussed in this paper. It is assumed that q, Q and p are positive real numbers satisfying $0<q<Q<1$ and $p>0$. Let (A) denote the class of functions $f(z)$ which are analytic and single-valued for $q \leqq |z| \leqq 1$ and satisfy $|f(z)| \leqq 1$ for $|z|=1$, $|f(z)| \leqq p$ for $|z|=q$. Let (B) denote that subclass of (A) for which the Laurent coefficients of its members are all nonnegative. Let (C) denote that subclass of (A) for which the members are analytic and single-valued throughout $|z| \leqq 1$. The first problem is the determination of $\text{l.u.b.}_{f\in(A)} |f(Q)|$ and the associated extremal functions. This problem was first solved by O. Teichmüller [Deutsche Math. 4, 16–22 (1939)] and subsequently by the author [Duke Math. J. 10, 341–354 (1943); these Rev. 4, 241]. The second problem is a modification of the first where the class (A) is replaced by the class (B). It was solved by F. Carlson [Ark. Mat. Astr. Fys. 26A, no. 9 (1938)]. The third problem, which was proposed to the reviewer by Walsh and was treated by the reviewer [Trans. Amer. Math. Soc. 55, 349–372 (1944); these Rev. 5, 259], is a modification of the first problem where the class (A) is replaced by the class (C). The principal results of these studies are stated and a brief account is given of the methods employed. *M. H. Heins* (Arlington, Va.).

Verjbinsky, M. On roots of a certain class of integral transcendental functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) 33, 99–102 (1941). [MF 9621]

The present paper, which aims to generalize some of the results of a preceding paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) 30, 781–784 (1941); these Rev. 2, 356], deals with real entire functions $f(x) = \sum_0^\infty a_k x^k$ of genus 0 and 1. Using as minor axis the line-segment joining each pair of conjugate imaginary zeros of $f(x)$, let us construct the ellipse whose major axis is $n^{\frac{1}{2}}$ $(n \geqq 1)$ times the minor axis and let us denote by R_n the ensemble of points in or on at least one of these ellipses. Then, according to Verjbinsky, if

$$P(x) = x(x-1) \cdots (x-n+2)(x-n+1+p), \quad p \geqq 0,$$

all the nonreal zeros of the entire function $F(x) = \sum_0^\infty a_k P(k) x^k$ lie in R_n. [It appears to the reviewer, however, that the author is not acquainted with the following more general result given by J. L. W. V. Jensen in Acta Math. 36, 181–195 (1913). If $g(x)$ is a polynomial of degree n with only real zeros, then all of the nonreal zeros of $g(D) \cdot f(x)$, where D is the derivative operator, lie in R_n. In fact, Verjbinsky's result follows from Jensen's theorem if we write

$$F(x) = x^{n-p} D(x^p D^{n-1} f(x))$$

and apply Jensen's theorem first to $D^{n-1} f(x)$ and then to $D(x^p D^{n-1} f(x))$. Also, the other theorems proved by Verjbinsky in the present paper seem to be special cases of theorems given in Jensen's paper.] *M. Marden.*

Levinson, Norman. The Gontcharoff polynomials. Duke Math. J. 11, 729–733 (1944). [MF 11575]

The Whittaker constant W is defined as the least upper bound of numbers c such that, if $f(z)$ is an entire function of exponential type c, and if $f(z)$ and each of its derivatives have at least one zero in the unit circle, then $f(z) \equiv 0$. By

use of the Gontcharoff polynomials the author obtains a new lower bound for W. It was pointed out by I. Kaplansky that the statement max $|H_3|=11/6$, which occurs on page 732, is incorrect, so that the value of the lower bound obtained here requires revision and should read .7199 instead of .7215. [The revised value was communicated by Levinson.] The best result hitherto was that of Boas, $W>.718$.
H. Pollard (New York, N. Y.).

Boas, R. P., Jr. Functions of exponential type. IV. Duke Math. J. 11, 799 (1944). [MF 11581]

[The first three notes appeared in the same J. 11, 9–15, 17–22, 507–511 (1944); these Rev. 5, 175; 6, 60.] The author shows that Whittaker's constant W [cf. note II] is less than .7399. The lower bound of .728 is a misprint and the best present value is .7199. *N. Levinson.*

Montel, Paul. Sur les différences divisées. C. R. Acad. Sci. Paris 215, 193–195 (1942). [MF 9487]

Announcement of the results published in Mathematica, Timişoara 19, 1–11 (1943); these Rev. 5, 92.

Strodt, Walter. Analytic solutions of non-linear difference equations. Ann. of Math. (2) 44, 375–396 (1943). [MF 8868]

The paper deals with the difference equation of degree n which is of the form

$$\sum_{k=1}^{m} a_k(x) \prod_{s=1}^{s_k} y(x+\omega_{k,s}) = \varphi(x), \tag{1}$$

in which the spans $\omega_{k,s}$ are arbitrary, the coefficients $a_k(x)$ are "almost constant" functions with finite limits at infinity and $\varphi(x)$ is such that, with some integer p, the function $x^{-p}\varphi(x)$ is "almost constant" and has a finite nonvanishing limit at infinity. A number of more detailed specifications are also imposed. It is proved that, for all sufficiently large positive constants D, M, the equation admits of precisely n solutions that are analytic in the half-plane $\Re(x)>D$ and that satisfy the relation $|y(x)|\leqq M|x|^{p/n}$. The method may be briefly described thus: an "approximating q-difference equation," of such form that it reduces to (1) as a certain parameter becomes infinite, is introduced. Limiting considerations applied to the solutions of this approximating equation yield the results stated. *R. E. Langer.*

Vekua, Ilja. On a new integral representation of analytic functions and its applications. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 2, 477–484 (1941). (Russian. Georgian summary) [MF 10293]

Vekua, Ilja. Supplements to the paper "On a new integral representation of analytic functions and its applications." Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 2, 701–706 (1941). (Russian. Georgian summary) [MF 10301]

The second paper gives a theorem which contains the result of the first one as a particular case. If

$$\varphi(z)=\varphi_0(z)+\sum_{k=1}^{m} P_k(z)\log(z-a_k),$$

where $\varphi_0(z)$ is holomorphic in a region T (bounded by $m+1$ closed curves) and belongs to class H_N (that is, as the variable approaches the boundary, the function and its first N derivatives each converge uniformly to functions satisfying Hölder conditions), where the $P_k(z)$ are polynomials of degree $N-1$ with real constant terms, and where the a_k are inside of each of the inside boundaries of T, then there exists a function $\mu(t)$ defined on the boundary L of T, satisfying a Hölder condition, and such that

$$\varphi(z)=\int_L \mu(t)(1-z/t)^{N-1}\log(1-z/t)ds+\int_L \mu(t)ds+ic,$$

where L is taken in the positive sense, ds is the element of arc on L, and c is a real constant. *S. Mandelbrojt.*

Vekua, Ilja. On a linear boundary value problem of Riemann. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 11, 109–139 (1942). (Russian. Georgian summary) [MF 10285]

Vekua, Ilja. Correction to the paper "On a linear boundary value problem of Riemann." Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 12, 215 (1943). (Russian) [MF 11690]

The author deals with the problem of determining a function $f(z)$ holomorphic in a bounded simply connected region, belonging in T to the class H_N [cf. the preceding review] and satisfying on the boundary L of T the condition

$$\Re\left(\sum_{k=0}^{N}\left[a_k(t)f^{(k)}(t)+\int_L h_k(t,t_1)f^{(k)}(t_1)ds_1\right]\right)=b(t),$$

$t\epsilon L$, $ds_1=$arc element of L, where the known functions $a_k(t)$, $h_k(t,t_1)$, $b(t)$ satisfy certain conditions. *S. Mandelbrojt.*

Picone, Mauro. Sulla definizione del logaritmo di una funzione olomorfa. Boll. Un. Mat. Ital. (2) 4, 6 pp. (1942) = Ist. Naz. Appl. Calcolo (2) no. 124. [MF 11730]

This paper states the following theorem. Let z_k be a complex variable situated in a circle A_k of the z_k-plane with center at ζ_k and let $Z=(z_1, z_2, \cdots, z_n)$ be the corresponding point of the $2n$-dimensional region $A=(A_1, A_2, \cdots, A_n)$. If the function $f(Z)$ is analytic in A and if $f(Z)\neq 0$ in A, then $f(Z)=f(Z_0)\exp\{g(Z)\}$, where Z_0 is the point $(\zeta_1, \cdots, \zeta_n)$ of A and $g(Z_0)=0$. The theorem is proved by the study of the possible holomorphic solutions of the partial differential equation

$$\sum(z_k-\zeta_k)\partial f(Z)/\partial z_k=[\mu+q(Z)]f(Z), \tag{1}$$

where $q(Z)$ is holomorphic in A and $q(Z_0)=0$. If μ is a positive or negative integer, a solution of (1) can be found which is holomorphic and nonvanishing in A, but such a solution does not exist if μ is not an integer. The theorem is also extended to provide for the representation $f(Z)=p(Z)\exp\{g(Z)\}$, where $p(Z)$ is an arbitrary polynomial. *M. Marden* (Milwaukee, Wis.).

Bochner, S. Group invariance of Cauchy's formula in several variables. Ann. of Math. (2) 45, 686–707 (1944). [MF 11375]

Let $f(z_1, \cdots, z_k)$ be analytic in a domain D. Then one form of Cauchy's formula states that

$$f(z_1, \cdots, z_k)=(1/(2\pi i)^k)\int_{C_k} f(\zeta_1, \cdots, \zeta_k)G(\zeta; z)d\omega_\zeta, \tag{1}$$

where $G(\zeta; z)=[(\zeta_1-z_1)\cdots(\zeta_k-z_k)]^{-1}$, $d\omega_\zeta=d\zeta_1\cdots d\zeta_k$ and C_k consists of the k-dimensional characteristic manifold $|\zeta_j-z_j|=r_j$, $j=1, \cdots, k$, the r_j being any positive numbers such that the point set $|\zeta_j-z_j|\leqq r_j$, $j=1, \cdots, k$, lies in D. From the standpoint of the paper under review what is important is not the fact that C_k has this particular form but that it is topologically equivalent to the torus $|\zeta_1|=\cdots=|\zeta_k|=1$ and in turn that the torus is invariant and transi-

tive under the group of transformations $\zeta_j \to \zeta_j e^{i\alpha_j}, j=1, \cdots, k$. The formula (1) has been altered in several ways by suitably modifying both $G(\zeta; z)$ and C_k. In this paper the author develops modifications which retain two main features, namely, a group invariance (with other groups of transformations) and the property that the kernel depends only upon the differences $\zeta_1-z_1, \cdots, \zeta_k-z_k$.

For the first theorem the author considers functions $f(z_1, \cdots, z_k)$ analytic in an octant-shaped tube

$$T\{(x_1, \cdots, x_k)\epsilon P; -\infty<y_j<\infty, j=1, \cdots, k\}$$

and of integrable square in T, that is,

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|f(x+iy)|^2 dv_y \leqq M_f$$

for (x) in P. (The point set P is called the basis of the tube T; T is called octant-shaped if (a) its basis is radiated, that is, if P is convex, does not contain the origin and contains with a point $x_1^0, \cdots, x_k^0$ all points $(tx_1^0, \cdots, tx_k^0)$, $0<t<\infty$, and if (b) its basis is of such a nature that its conjugate set G is an open radiated set for which the integral

$$\int_G \exp\{-\alpha_1x_1-\cdots-\alpha_kx_k\}dv_\alpha$$

converges boundedly in every compact subset of P. The set G conjugate to a set P consists of all points $\alpha_1, \cdots, \alpha_k$ for which $\alpha_1x_1+\cdots+\alpha_kx_k>0$ holds for all x in P.) The author isolates a k-dimensional portion of the boundary of T, namely, the spine $\{\zeta_j=0+i\eta_j; -\infty<\eta_j<\infty, j=1, \cdots, k\}$. Because of formal advantages he uses the oriented external differential $d\zeta_1\cdots d\zeta_k=i^k d\eta_1\cdots d\eta_k$ instead of the Euclidean volume element $dv_\eta=d\eta_1\cdots d\eta_k$. Using this differential he denotes the spine as a manifold by iE_k. The first theorem is as follows. If T is an octant-shaped tube and if $f(z_1, \cdots, z_k)$ is analytic and of integrable square in T then

$$f(z)=(2\pi i)^k\int_{iE_k}K(z-\zeta)f(\zeta)d\zeta_1\cdots d\zeta_k,$$

where $K(z)$ is the kernel

$$\int_G \exp\{-(z_1\alpha_1+\cdots+z_k\alpha_k)\}dv_\alpha.$$

The proof depends upon Fourier transforms. The simplest case occurs when P is the octant $x_1>0, \cdots, x_k>0$ which is self-conjugate. For this case the kernel $K(z)$ is $(z_1\cdots z_k)^{-1}$.

Next let $k=n^2$. Then the variables $z_1, \cdots, z_k$ can be re-indexed into a square scheme z_{pq}, $p, q=1, \cdots, n$. The space of these variables is called the nonsymmetric matrix space. If the matrix is symmetric ($z_{pq}=z_{qp}$) then there are $k=n(n+1)/2$ independent variables and the space is called symmetric. Theorem 2 gives Cauchy's formula for the symmetric case for functions of integrable square over a tube T_n^0, the basis P_n^0 of T_n^0 consisting of those matrices (x) whose roots are all strictly positive. Theorem 3 treats the nonsymmetric case. In part II the author transfers the formulas of theorems 2 and 3 from tubes to a type of unit sphere, and in part III he makes an analogous investigation for the general formula of theorem 1. *W. T. Martin.*

Bochner, S. Boundary values of analytic functions in several variables and of almost periodic functions. Ann. of Math. (2) 45, 708–722 (1944). [MF 11376]

A classical theorem due to F. Riesz states that, if $f(z)$ is analytic in $|z|<1$ and if $\int_0^{2\pi}|f(re^{i\theta})|d\theta\leqq\gamma<\infty$ for $0<r<1$, then there exists a boundary function $f(e^{i\theta})$ such that $\int_0^{2\pi}|f(re^{i\theta})-f(e^{i\theta})|d\theta\to 0$ as $r\to 1$. The usual proof is based upon the Blaschke decomposition of $f(z)$. Hardy and Littlewood have sharpened Riesz's theorem so that the same hypotheses yield the conclusion that

$$\int_0^{2\pi}\sup_{0<r<1}|f(re^{i\theta})|d\theta\leqq\alpha\cdot\gamma,$$

where α is a universal constant. By giving a qualified extension of this result to Bohr almost periodic functions the author succeeds in extending Riesz's result to functions of several variables. The extension cannot be carried out directly since no decomposition comparable to the Blaschke one exists for several variables. The author uses the results of Hardy and Littlewood to obtain similar results for analytic functions $f(x+iy)$ in the half plane $0<x<\infty$. The author then uses these results to prove the following theorem. Theorem 3. If $f(z)$ is an analytic Bohr almost periodic function in $x>0$ and if $M_y|f(x+iy)|<\gamma$ $(0<x<\infty)$, then for each $x_0>0$ we have

$$M_y\{\sup_{0\leqq x\leqq\infty}|f(x+iy)|\}\leqq\alpha\cdot\gamma.$$

Using this result and a customary connection between periodic functions of several variables and almost periodic functions of one variable, he derives his main theorem. Theorem 4. If a power series

$$f(s_1, \cdots, s_k)=\sum_{m_1, \cdots, m_k=0}^{\infty}a_{m_1\cdots m_k}\exp\{-(m_1s_1+\cdots+m_ks_k)\},$$

$s_j=\sigma_j+it_j$, is absolutely convergent in the octant $\sigma_1>0, \cdots, \sigma_k>0$ and if for all σ in the octant we have

$$(*)\qquad \int_0^{2\pi}\cdots\int_0^{2\pi}|f(\sigma+it)|dt_1\cdots dt_k<\gamma,$$

then the limiting series

$$\sum a_{m_1\cdots m_k}\exp\{-(m_1t_1+\cdots+m_kt_k)\}$$

is the Fourier series of a function of class L^1. More generally, if a Laurent series

$$f(s_1, \cdots, s_k)=\sum_{-\infty}^{\infty}a_{m_1\cdots m_k}\exp\{-(m_1s_1+\cdots+m_ks_k)\}$$

is absolutely convergent in a k-dimensional domain P of the σ-space and if P is the union of half lines issuing from but not containing the origin and if relation (*) holds in P, then the limiting series is the Fourier series of a function in L^1.

The author obtains other results including boundary value theorems for Besicovitch almost periodic functions.

W. T. Martin (Syracuse, N. Y.).

Hua, Loo-Keng. On the theory of automorphic functions of a matrix variable. I. Geometrical basis. Amer. J. Math. 66, 470–488 (1944). [MF 10916]

Hua, Loo-Keng. On the theory of automorphic functions of a matrix variable. II. The classification of hypercircles under the symplectic group. Amer. J. Math. 66, 531–563 (1944). [MF 11394]

These papers are geometrical and group-theoretical in nature. The first paper gives a rather condensed survey of the author's investigations, which extend those of Siegel on "symplectic geometry" [Amer. J. Math. 65, 1–86 (1943); these Rev. 4, 242] to two other groups. Let italic capital

letters denote $n\times n$ matrices with complex elements; let I denote the unit matrix and O the zero matrix; moreover, let German capital letters refer to $2n\times 2n$ matrices. The author defines then the special matrices

$$\mathfrak{F}=\begin{pmatrix} O & I \\ -I & O \end{pmatrix}, \quad \mathfrak{F}_1=\begin{pmatrix} O & I \\ I & O \end{pmatrix}.$$

For matrices

$$\mathfrak{T}=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

he defines 3 groups. (I) The $\mathfrak{T}$ satisfying $\mathfrak{T}\mathfrak{F}\mathfrak{T}'=\mathfrak{F}$ form the symplectic group. (II) The $\mathfrak{T}$ satisfying $\mathfrak{T}\mathfrak{F}_1\mathfrak{T}'=\mathfrak{F}$ form the orthogonal group. (III) The $\mathfrak{T}$ satisfying $\bar{\mathfrak{T}}\mathfrak{F}\mathfrak{T}'=\mathfrak{F}$ form the conjunctive-symplectic group.

The spaces in which these transformations are performed are introduced first in "homogeneous coordinates," which means that two pairs (Z_1, Z_2) and (QZ_1, QZ_2) for any nonsingular Q are identified as the same point. For the symplectic group, for example, only symmetric nonsingular pairs of matrices are to be considered, a pair (Z_1, Z_2) of matrices being called symmetric if $Z_2Z_1'=Z_1Z_2'$ or $(Z_1, Z_2)\mathfrak{F}(Z_1, Z_2)'=O$, and nonsingular if the $n\times 2n$ matrix (Z_1, Z_2) is of rank n. Such a space under the symplectic group would be analogous to the projective space. In order now to single out a subgroup one can choose a Hermitian matrix $\mathfrak{H}$ and consider those nonsingular symmetric pairs (Z_1, Z_2) for which $(\overline{Z_1, Z_2})\mathfrak{H}(Z_1, Z_2)'$ is the matrix of a positive definite Hermitian form. These pairs (or points) form a "hypercircle." The group of motions is here the subgroup of those $\mathfrak{T}$ which leave $\mathfrak{H}$ invariant. Here a symplectic classification of Hermitian matrices is required. Those which are equivalent to

$$\begin{pmatrix} I & O \\ O & -I \end{pmatrix}$$

lead to a "hyperbolic" space, which corresponds to the case treated by Siegel. Similar specializations are proposed for the other groups. The author then introduces nonhomogeneous coordinates and uses an invariant quadratic differential form for a metrization and for the definition of geodesics. These investigations are parallel to those of Siegel, however more general.

The second paper is concerned with the symplectic group only. Calling two symmetric pairs of matrices (A_1, B_1) and (A, B) equivalent under the symplectic group if a symplectic $\mathfrak{T}$ and a nonsingular Q exist such that

$$(A_1, B_1)=Q(A, B)\mathfrak{T},$$

the author proves first that any two nonsingular symmetric pairs of matrices are equivalent. Hypercircles have been defined in the previous paper, $\mathfrak{H}$ being called the matrix of the hypercircle. The main object of this paper is the classification of hypercircles according to "conjunctivity under the symplectic group." Two $2n\times 2n$ Hermitian matrices $\mathfrak{H}$ and $\mathfrak{H}_1$ are called conjunctive if there exists a symplectic $\mathfrak{T}$ such that $\mathfrak{H}_1=\bar{\mathfrak{T}}\mathfrak{H}\mathfrak{T}'$. The skew-symmetric matrix $\mathfrak{D}(\mathfrak{H})=\mathfrak{H}'\mathfrak{F}\mathfrak{H}$ is called the discriminantal matrix of the hypercircle. Conjunctivity of $\mathfrak{H}$ and $\mathfrak{H}_1$ implies congruence of $\mathfrak{D}(\mathfrak{H})$ and $\mathfrak{D}(\mathfrak{H}_1)$ under the symplectic group. The problem of classification is simplified by the fact that every $\mathfrak{H}$ is conjunctive to a special one of the form

$$\begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix}$$

("binomial hypercircle"). The author finds it necessary to supplement the existing theory of pairs of Hermitian matrices. Besides their elementary divisors he introduces certain "signatures" belonging to the real roots in order to get necessary and sufficient conditions for conjunctivity. The signatures of negative and of nonnegative roots behave differently in the classification. Applying these results the author comes to a canonical form of hypercircles such that each conjunctivity class contains only one canonical hypercircle. The paper contains quite a number of bothersome misprints. *H. Rademacher* (Swarthmore, Pa.).

Min, Szu-Hoa. Non-analytic functions. Amer. Math. Monthly **51**, 510–517 (1944). [MF 11476]

The author establishes several results, some old and some new, in the theory of polygenic functions. The following generalization of Cauchy's integral theorem is given. If $f(z)$ is of nonanalyticity r in a simply connected region R and if C is a rectifiable simple closed curve lying entirely within R, then

$$\left|\int_C f(z)dz\right| \leqq 4\sqrt{2}\Omega r,$$

where Ω is the area enclosed by C. Bibliographical notes by J. DeCicco are added. *E. F. Beckenbach* (Austin, Tex.).

Fourier Series and Generalizations, Integral Transforms

Natanson, I. P. Sur le critère de Dini dans la théorie d'interpolation trigonométrique. C. R. (Doklady) Acad. Sci. URSS (N.S.) **42**, 53–56 (1944). [MF 11625]

A statement of results published in the meantime [Ann. of Math. (2) **45**, 457–471 (1944); these Rev. **6**, 48]. *A. Zygmund* (South Hadley, Mass.).

Bellman, Richard. A note on a theorem of Hardy on Fourier constants. Bull. Amer. Math. Soc. **50**, 741–744 (1944). [MF 11288]

Given two sequences of numbers $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$, let $A_n=\sum_1^n a_\nu/n$, $B_n=\sum_n^\infty b_\nu/\nu$ (provided $\sum b_\nu/\nu$ converges). Hardy proved [Messenger of Math. **58**, 50–52 (1928)] that, if $f(x)\sim\sum_1^\infty a_\nu \cos \nu x \in L^p$, $p\geqq 1$, then $\sum_1^\infty A_\nu \cos \nu x$ is the Fourier series of an $F\in L^p$. The author now shows that, if $g(x)=\sum_1^\infty b_\nu \cos \nu x \in L^p$, $p>1$, then $\sum_1^\infty B_\nu \cos \nu x$ is the Fourier series of a $G(x)\in L^p$. (If $p=1$, the series $\sum b_\nu/\nu$ need not converge.) *A. Zygmund* (South Hadley, Mass.).

Szász, Otto. The generalized jump of a function and Gibbs' phenomenon. Duke Math. J. **11**, 823–833 (1944). [MF 11583]

Suppose that $f(x)$ is integrable, of period 2π, and odd. Let $f(x)\sim\sum b_n \sin nx$. If for some $\alpha>0$ $\lim_{t\to 0}\psi_\alpha(t)=j/2$, where

$$\psi_\alpha(t)=\alpha t^{-\alpha}\int_0^t (t-u)^{\alpha-1}f(u)du$$

exists, the number j is called the generalized jump of f at the point $x=0$. The author investigates the influence of a generalized jump on the Gibbs phenomenon and on the summability of the sequence nb_n. The following theorem may serve as an example of the results obtained. Suppose that

$$2\psi_{\alpha+1}(t)\to j, \quad \int_0^t |2\psi_\alpha(u)-j|du=O(t), \qquad t\downarrow 0,$$

for some $\alpha<1$. Then

$$n^{-1}\sum_{1}^{n} b_\gamma \sin \gamma t_n \to 2\pi^{-2}(-1)^{\lambda-1}j/(2\lambda-1)$$

for $nt_n-\frac{1}{2}(2\lambda-1)\pi=O(n^{-1})$, $t_n\downarrow 0$; $\lambda=1, 2, \cdots$.

A. Zygmund (South Hadley, Mass.).

Szász, Otto. On uniform convergence of trigonometric series. Bull. Amer. Math. Soc. **50**, 856–867 (1944). [MF 11554]

Let $\phi(t)$ be even, $f(t)$ odd, both integrable and of period 2π. Let

$$\phi(t)\sim a_0/2+\sum_{1}^{\infty} a_n \cos nt, \quad f(t)\sim \sum_{1}^{\infty} b_n \sin nt.$$

(i) Under the assumption

$$\text{(S)} \qquad \lim_{\lambda\downarrow 1} \limsup_{n\to\infty} \sum_{n}^{\lambda n} (|a_\nu|-a_\nu)=0,$$

the series $\sum a_n \cos nt$ converges uniformly at every point of continuity of ϕ. (ii) If $\{b_n\}$ satisfies condition (S), $\sum b_n \sin nt$ converges uniformly at every point of continuity of f. (iii) Suppose that

$$\sum_{n}^{2n}|b_\nu-b_{\nu+1}|=O(n^{-1})$$

and that $\sum a_n$ is Abel summable; then $\sum a_n \cos nt$ is uniformly convergent. A similar result holds for power series. An extension is also given of a familiar result of Chaundy and Jolliffe asserting that, if $b_n \geqq b_{n+1}>0$, $nb_n\to 0$, then $\sum b_n \sin nt$ converges uniformly. *A. Zygmund.*

Loo, Ching-Tsün. Note on the strong summability of Fourier series. Trans. Amer. Math. Soc. **56**, 519–527 (1944). [MF 11494]

The paper is devoted to the following extension of a well-known result of Hardy and Littlewood. If the positive integer k is smaller than p and if

$$\int_0^t |\phi(u)|^p du=o(t),$$

where $2\phi(u)=f(x+u)+f(x-u)-2f(x)$ and $f(x)$ is periodic and belongs to L^p, then

$$\sum_{m=0}^{n} |S_{m(k)}(x)-f(x)|^2=o(n),$$

where $S_q(x)$ is the partial sum of order q of the Fourier series of $f(x)$, and $m(k)=m^k$. *R. Salem.*

Loo, Ching-Tsün. Two Tauberian theorems in the theory of Fourier series. Trans. Amer. Math. Soc. **56**, 508–518 (1944). [MF 11493]

The purpose of the paper is to prove the following generalization of a result of Hardy and Littlewood [Ann. Scuola Norm. Super. Pisa (2) **3**, 43–62 (1934)]. Let $f(x)$ be periodic and of class L and let $\sigma_n{}^\alpha(x)$ be its nth Cesàro sum of order $\alpha>0$. Write

$$2\phi(t)=f(x+t)-2f(x);$$

$$\phi_p(t)=(p/t^p)\int_0^t (t-u)^{p-1}\phi(u)du.$$

Then (1) if $\sigma_n{}^\alpha(x)-f(x)=o(1/\log n)$, $\phi_{1+\alpha}(t)=o(1)$; (2) if $\sigma_n{}^\alpha(x)-f(x)=O(n^{-\epsilon})$ $(0<\epsilon<1)$ and the Fourier coefficients of f are $O(n^{-\delta})$ with $\delta>1-\epsilon$, then $\phi_\alpha(t)=o(1)$.

R. Salem (Cambridge, Mass.).

Herriot, John G. Nörlund summability of multiple Fourier series. Duke Math. J. **11**, 735–754 (1944). [MF 11576]

Let $f(x_1, x_2, \cdots, x_k)$ be an L-integrable function, of period 2π with respect to each variable, and let

$$f\sim \sum c_{n_1,\cdots,n_k} \exp\{i(n_1x_1+\cdots+n_kx_k)\}.$$

Let

$$S_N(x_1, \cdots, x_k)=\sum c_{n_1,\cdots,n_k} \exp\{i(n_1x_1+\cdots+n_kx_k)\},$$

$|n_1|\leqq N, \cdots, |n_k|\leqq N$, be the Nth "cubic" partial sum of the Fourier series. It is classical that for $k=1$ the behavior of S_N at a point depends only on the values of f in the neighborhood of that point ("localization property"). This is no longer true for $k=2$, but, as is shown here, the localization property remains valid for the (C, α) means of the sequence $\{S_N\}$, provided $\alpha\geqq 1$. For $\alpha<1$ this result does not hold. If $k\geqq 3$, the localization property does not hold even if $\{S_N\}$ is summed by Abel's method (a fortiori, it does not hold for any Cesàro method). These results are also extended to Nörlund's method of summation and to the "triangular" partial sums

$$T_N=\sum c_{n_1,\cdots,n_k} \exp\{i(n_1x_1+\cdots+n_kx_k)\},$$

$|n_1|+\cdots+|n_k|\leqq N$. *A. Zygmund.*

Cooper, J. L. B. The uniqueness of trigonometrical integrals. Proc. London Math. Soc. (2) **48**, 292–309 (1944). [MF 11386]

With functions $f(u)$, integrable in every finite interval, functions $F(t)$ are associated by

$$\text{(1)} \qquad F(t)=\int_{-\infty}^{\infty} f(u)e^{iut}du,$$

where the integral is to be taken in an appropriately generalized sense. The problem of uniqueness is the problem of whether, with a specified meaning for the integral in (1), two f's with the same F are necessarily equal almost everywhere. In particular, there will be uniqueness if

$$\text{(2)} \qquad f(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{-iut}dt,$$

with an appropriately generalized integral. It was proved by Offord [Proc. London Math. Soc. (2) **42**, 422–480 (1937)] that there is uniqueness in (1) if the integral is summable $(C, 1)$ and two F's are considered the same only if they coincide for every t; the order of summability is the largest possible, and a single exceptional point destroys the theorem. On the other hand, Macphail and Titchmarsh [J. London Math. Soc. **11**, 313–318 (1936)] showed that if the integral in (1) is summable uniformly (C, n) then (2) is true $(C, n+1)$. The author generalizes this result by replacing uniform convergence by mean convergence. His main result is that if the (C, n) "partial sums" of (1) converge in mean, of order 1, over any finite interval, then (2) is summable $(C, n+3)$ almost everywhere, and $(C, n+1)$ if $\int f(u)du$ is summable (C, n). Methods of summability other than (C, n) are also discussed. *R. P. Boas, Jr.*

(for $n=0, 1$)

Pérès, Joseph. Calcul symbolique d'Heaviside et calcul de composition de Vito Volterra. C. R. Acad. Sci. Paris **217**, 517–520 (1943). [MF 11669]

The author compares the techniques of the Heaviside operational calculus with that of Volterra's calculus of composition [Volterra and Pérès, Leçons sur la Composition et

les Fonctions Permutables, Gauthier-Villars, Paris, 1924]. It is noted that the Heaviside calculus is contained within Volterra's theory as a special case. *A. E. Heins.*

Pérès, Joseph. Quelques applications du calcul de composition de Volterra. C. R. Acad. Sci. Paris **217**, 585–588 (1943). [MF 11673]

An example is given to illustrate the remarks made by the author in the paper reviewed above. *A. E. Heins.*

Grünberg, G. A. Relation between operational expressions of two arbitrary functions and the operational representation of their product. C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 141–143 (1943). [MF 11167]

The writer presents a theorem on the composition of a pair of Laplace transforms. The theorem is not new [cf. theorem 2, Bourgin-Duffin, Amer. J. Math. **59**, 489–505 (1937)]. *D. G. Bourgin* (Urbana, Ill.).

Hadwiger, H. Ein transzendentes Additionstheorem und die Neumannsche Reihe. Mitt. Verein. Schweiz. Versich.-Math. **42**, 57–66 (1942). [MF 11449]

The author compiles sixteen pairs of Laplace transforms $\phi(s)$, $\psi(s)$ for which the function $\Phi(x)$ with the Laplace transform $\phi^n(s)\psi^m(s)$ can be written down explicitly. No novelty is claimed and ample references to literature are given. The table may be useful in connection with the integral equation of renewal theory

$$F(x)=G(x)+\int_0^x \Phi(x-\xi)F(\xi)d\xi.$$

The reviewer does not understand the relevance of the title. *W. Feller* (Providence, R. I.).

Garabedian, H. L. The analogue of Bromwich's theorem for integral transformations. Ann. of Math. (2) **45**, 740–746 (1944). [MF 11379]

Let n be a positive integer. Conditions on $k(x, t)$ are given which ensure that the transform

$$z(x)=\int_0^\infty k(x, t)a(t)dt$$

exists for $x>0$ and converges to l as $x\to\infty$ whenever $a(u)$ is bounded and Lebesgue integrable over each finite interval $0\leqq u\leqq A$ and such that the function $\int_0^x a(u)du$ is (as $x\to\infty$) summable by the Cesàro method C_n. *R. P. Agnew.*

Haviland, E. K. A note on the Lambert transform. Amer. J. Math. **66**, 523–530 (1944). [MF 11393]

Let $\alpha(x)$ be of bounded variation in every finite interval $(0, b)$, and constant near $x=0$, and write

$$A(s)=\int_0^\infty e^{-sx}d\alpha(x), \quad L(s)=\int_0^\infty (xe^{-sx}/(1-e^{-sx}))d\alpha(x).$$

The author proves that the integrals converge or diverge together, and that, if $sL(s)\to 0$ as $s\to 0+$, then $A(s)\to 0$ as $s\to 0+$. This is the integral analogue of a theorem of Hardy and Littlewood [Proc. London Math. Soc. (2) **19**, 21–29 (1921)] for power series; because the integrals may fail to converge absolutely or uniformly, the original proof requires considerable modification. However, the prime number theorem is applied in the same way as in the work of Hardy and Littlewood. *R. P. Boas, Jr.* (Cambridge, Mass.).

Hirschman, I. I., Jr. Two power series theorems extended to the Laplace transform. Duke Math. J. **11**, 793–797 (1944). [MF 11580]

The author proves for Laplace integrals the analogues of the Ostrowski and Jentzsch theorems for power series. *H. Pollard* (New York, N. Y.).

Pollard, Harry. On Stieltjes' integral equation. Ann. of Math. (2) **46**, 83–87 (1945). [MF 11790]

The convergence of the integral

$$(1) \qquad f(x)=\int_0^\infty d\alpha(t)/(x+t)$$

implies

$$(2) \qquad f(x)=\int_{0+}^\infty e^{-xu}du\int_0^\infty e^{-tu}d\alpha(t)$$

and (3) $\alpha(t)=o(t)$ $(t\to\infty)$. In the present paper the author proves that (2) and (3) imply (1). Also a set of necessary and sufficient conditions that $f(x)$ have the representation (1), where $\alpha(t)$ is of bounded variation in $(0, R)$ for every $R>0$, is given in terms of Widder's operator

$$L_{k,t}[f]=\frac{(-t)^{k-1}}{k!(k-2)!}\frac{d^{2k-1}}{dt^{2k-1}}[t^kf(t)], \quad k=2, 3, \cdots.$$

F. G. Dressel (Durham, N. C.).

Pollard, Harry. Completeness theorems of Paley-Wiener type. Ann. of Math. (2) **45**, 738–739 (1944). [MF 11378]

A theorem of Paley and Wiener [Fourier Transforms in the Complex Domain, Amer. Math. Soc. Colloquium Publ., vol. 19, New York, 1934, p. 100] contains, in particular, the result that sequences $\{x_n\}$ and $\{y_n\}$ in Hilbert space are both complete or both incomplete if, for every finite sequence of numbers $\{a_n\}$,

$$\|\sum a_n(x_n-y_n)\|^2\leqq\lambda\|\sum a_nx_n\|^2, \qquad 0\leqq\lambda<1.$$

The author shows that this conclusion remains true if the right side is replaced by

$$\lambda_1\|\sum a_nx_n\|^2+\lambda_2\|\sum a_ny_n\|^2,$$

where $0\leqq\lambda_1<1$ and $0\leqq\lambda_2<1$; or by

$$\lambda\{\|\sum a_nx_n\|+\|\sum a_ny_n\|\}^2, \qquad 0\leqq\lambda<\tfrac{1}{4}.$$

If the sets are both orthonormal, the condition becomes

$$\sum\sum a_m\bar{a}_n(x_m, y_n)\geqq\mu\sum|a_n|^2, \qquad \mu>0.$$

R. P. Boas, Jr. (Cambridge, Mass.).

Markouchevitch, A. Sur les critères pour qu'un système de fonctions analytiques soit complet. C. R. (Doklady) Acad. Sci. URSS (N.S.) **43**, 3–6 (1944). [MF 11606]

A set $\{u_n(z)\}$ of analytic functions, regular in $|z|<1$, is said to be complete if, for each $r<1$, every $f(z)$ which is regular in $|z|\leqq r$ is the uniform limit of a sequence of linear combinations of the $u_n(z)$. The author states two general criteria for completeness and a number of special cases, of which we mention two. Theorem 4. The set $\{F(z\zeta_n)\}$ is complete in $|z|<1$ if $F(z)$ is an entire function of order μ and type σ, $F^{(n)}(0)\neq 0$ $(n=0, 1, 2, \cdots)$, and

$$e\mu\sigma<\limsup_{n\to\infty} n|\zeta_n|^{-\mu};$$

this and others of his theorems generalize results of A. Gelfond [Rec. Math. [Mat. Sbornik] N.S. **4(46)**, 149–156 (1938)]. The author's method for these theorems appears

to consist in showing that

$$H(z)=\int_0^{2\pi} F(ze^{i\theta})h(\theta)d\theta, \qquad h(\theta)\epsilon L^2,$$

vanishes identically if $H(\zeta_n)=0$; and so the problem is reduced to the problem of the distribution of the zeros of an entire function. Theorem 6. If $F(z)$ is analytic in $|z|<1$ and $F^{(n)}(0)\neq 0$ $(n=0, 1, 2, \cdots)$, then the set $\{z^n F_n(z)\}$, where $F_0(z)=F(z)$ and $F_n(z)=\int_0^z F_{n-1}(w)dw$, is complete in $|z|<1$. The author's theorem 5 appears to be incorrect as stated; since it contains misprints, the reviewer is not sure what its content is intended to be. *R. P. Boas, Jr.*

Nikolsky, S. Approximation par polynômes des fonctions vérifiant la condition de Lipschitz. C. R. (Doklady) Acad. Sci. URSS (N.S.) **42**, 108–111 (1944). [MF 11627]

Given a number $M>0$, let MH denote the class of functions $f(x)$, $-1\leqq x\leqq +1$, such that $|f(x')-f(x'')|\leqq M|x'-x''|$ for all x', x''. Let $a_0/2+\sum_{\nu=1}^{\infty} a_\nu \cos \nu\theta$ be the Fourier series of the function $f(\cos\theta)$, and let $S_n(\theta)$ be the $(n-1)$st partial sum of that series. Given any set $\{\eta\}$ of real numbers $\eta_1, \eta_2, \cdots, \eta_{n-1}$, we set

$$U_n(\eta; f, \theta)=a_0/2+\sum_{\nu=1}^{n-1} a_\nu \eta_\nu \cos \nu\theta,$$

$$E_n(\eta; MH)=\sup_{f\epsilon MH}\sup_{\theta}|f(\cos\theta)-U_n(\eta; f, \theta)|.$$

It is shown that (i) for any system $\{\eta\}$ of numbers $\eta_1, \eta_2, \cdots, \eta_{n-1}$,

$$E_n(\eta; MH)\geqq M\pi/(2n)+\epsilon_n,$$

where ϵ_n does not depend on $\{\eta\}$ and is $O(n^{-2})$ for $n\to\infty$; (ii) if $\eta_\nu=(\nu\pi/(2n))\cot(\nu\pi/(2n))$, $\nu=1, 2, \cdots, n-1$, then $E_n(\eta; MH)<M\pi/(2n)$. [Part (ii) is essentially in Favard, Bull. Sci. Math. (2) **61**, 209–224, 243–256 (1937).]

A. Zygmund (South Hadley, Mass.).

Ghizzetti, Aldo. Sui momenti di una funzione limitata. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **77**, 11 pp. (1942) = Ist. Naz. Appl. Calcolo (2) no. **122**. [MF 11517]

Ghizzetti, Aldo. Ricerche sui momenti di una funzione limitata compresa fra limiti assegnati. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. (7) **13**, 1165–1199 (1942) = Ist. Naz. Appl. Calcolo (2) no. 141. [MF 11515]

Consider a continuous function $f(x)$ in $-\infty\leqq a<x<b\leqq\infty$, and suppose that $0\leqq f(x)\leqq 1$. Let μ_k be the kth moment of $f(x)$. The problem considered is: given $\mu_0, \cdots, \mu_{k-1}$, to find best inequalities for μ_k. For the solution the author introduces "rectangular functions of order n," that is, functions which equal 1 on n subintervals of (a, b) and vanish otherwise. In the first paper the problem is solved by elementary computations for the interval $(0, 1)$ and $k=1, 2$. The second paper takes up the general problem and shows that it can be reduced to the case of rectangular functions. These are treated by algebraic means, using Hankel determinants of two sequences of numbers, called "bimomenti" and related to the μ_k by linear recursive formulas. The connection with the Hausdorff solution is analyzed. In a postscript the author states that the main results have also been obtained, by different methods, by N. Achyèser and M. Krein [Comm. Soc. Math. Kharkoff et Inst. Sci. Math. Méc. Univ. Kharkoff (4) **12**, 13–33 (1935)], Verblunsky [Proc. Cambridge Philos. Soc. **32**, 30–39 (1936)] and L. Kantorovič [C. R. (Doklady) Acad. Sci. URSS (N.S.) **14**, 531–537 (1937)].

W. Feller (Providence, R. I.).

Ghizzetti, Aldo. Sui momenti di 2° ordine di una legge di probabilità in n dimensioni. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **4**, 94–101 (1943) = Ist. Naz. Appl. Calcolo (2) no. **152**. [MF 11522]

Let $x=(x_1, \cdots, x_n)$, $dT=dx_1\cdots dx_n$, and denote by m_{ik} the elements of an arbitrarily prescribed symmetric n by n matrix. The following theorem is proved. In order that there exist a continuous $f(x)$ such that $0\leqq f\leqq L$ and

$$\int fdT=0, \quad \int x_i fdT=0, \quad \int x_i x_k fdT=m_{ik},$$

it is necessary and sufficient that the quadratic form $\sum m_{ik}\xi_i\xi_k$ be positive definite and that its discriminant D satisfy the inequality

$$DL^2(n+2)^n\pi^n\geqq\{\Gamma((n+2)/2)\}^2.$$

The equality holds only for an $f(x)$ which has the constant value L within the ellipsoid $\sum M_{ik}x_i x_k=n+2$, where (M_{ik}) is the inverse matrix of (m_{ik}). *W. Feller.*

Calculus of Variations

Botts, Truman. Sufficient conditions for a generalized-curve problem in the calculus of variations. Duke Math. J. **11**, 373–403 (1944). [MF 10676]

The present paper is concerned with a sufficiency theorem for the free generalized curve problem in nonparametric form in $(n+1)$-space. Using a rather restrictive definition of a weak neighborhood, the author shows that the analogues of the DuBois Reymond, Legendre and Jacobi necessary conditions hold and that these conditions when suitably strengthened are sufficient to insure a weak relative minimum. This appears to be the first sufficiency theorem of this type on a class of generalized curves. As stated by the author, free use is made of the ideas of L. C. Young [Acta Math. **39**, 229–258 (1938)] and E. J. McShane [Duke Math. J. **7**, 1–27 (1940) and Trans. Amer. Math. Soc. **52**, 344–379 (1942); these Rev. **2**, 226 and **4**, 48]. In particular, the sufficiency proof is a modification of an indirect proof introduced by E. J. McShane. *M. R. Hestenes.*

Wang, Hsien-Chung. On the paths with Monge's equations of the second degree as conditions of intersection. Bull. Amer. Math. Soc. **50**, 935–942 (1944). [MF 11567]

A set of invariants of a system of paths in a space of $n+1$ dimensions is considered. Their vanishing signifies geometrically that the conditions of intersection of neighboring paths are given by a system of Monge's equations of the second degree. These invariants play an important rôle in Douglas's treatment of the inverse problem of the calculus of variations and, in particular, in his solution of the problem in the three-dimensional case. *S. Chern.*

Lusternik, L. On the number of solutions of a variational problem. C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 215–217 (1943). [MF 11163]

The present paper extends to the case of variational problems the results of previous papers [C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 59–61 (1943); **39**, 88–90 (1943); these Rev. **5**, 273] concerning functions of a finite number of variables. It is concerned with a topological property of critical sets on a compactum. An application to geodesics yields the following result. The lengths of geodesics $c_1\leqq c_2\leqq c_3\leqq\cdots$ joining any two points a and b on a surface

of genus 0 satisfy the exact inequalities $c_m < c_{m+2}$ for every m. If $c_m = c_{m+1} = c$ for even m, then every geodesic passing through a passes through b too, so that the length of the arc of the geodesic cut out by the points a and b is c. The c_m's in this theorem are defined by means of a minimax principle. *M. R. Hestenes* (Chicago, Ill.).

Cox, Mary Jane. On necessary conditions for relative minima. Amer. J. Math. 66, 170–198 (1944). [MF 10567]

During the last decade attempts have been made to establish necessary conditions and sufficient conditions for a minimum in the problem of Bolza without assumptions of normality. Prior to the present paper necessary conditions for a minimum have been established by Graves and McShane free from normality assumptions except in regard to the nonnegativeness of the second variation. The present paper establishes a theorem, conjectured by McShane, stating that, if an arc C_0 is a minimizing arc, then, for every admissible variation η satisfying the variational side conditions, there exist multipliers $\lambda_0 \geqq 0$, $\lambda_\beta(t)$ with which C_0 satisfies the Euler equation, the transversality condition, the Weierstrass condition, the Clebsch condition and for which the second variation $J_2(\eta, \lambda)$ is nonnegative. Thus necessary conditions have been established completely without normality assumptions. The corresponding sufficiency theorem has been established by E. J. McShane [Trans. Amer. Math. Soc. 52, 344–379 (1942); these Rev. 4, 48] for a weak relative minimum and by F. G. Myers [Duke Math. J. 10, 73–97 (1943); these Rev. 4, 200] for a semi-strong relative minimum. *M. R. Hestenes* (Chicago, Ill.).

Shiffman, Max. Instability for double integral problems in the calculus of variations. Ann. of Math. (2) 45, 543–576 (1944). [MF 10923]

The writer considers problems in parametric form in which

$$J(\zeta) = \iint_C [f(X, Y, Z) + k(X^2+Y^2+Z^2)^{\frac{1}{2}}]dudv,$$

where $\zeta(u, v)$ is absolutely continuous in the sense of Tonelli on the unit circle C with finite Dirichlet integral

$$D(\zeta) = \tfrac{1}{2}\iint (\zeta_u{}^2 + \zeta_v{}^2)dudv$$

and X, Y and Z are the usual Jacobians $(y_u z_v - y_v z_u)$, etc., x, y and z being the components of ζ. Let f be positively homogeneous of the first degree and convex in its arguments, of class C' if $(X, Y, Z) \neq (0, 0, 0)$, and let $0 < m \leqq f \leqq k' < k$ for $X^2+Y^2+Z^2=1$. The writer denotes the integral

$$\iint_C f(X, Y, Z)dudv + kD(\zeta)$$

by $I(\zeta)$; $I(\zeta) \geqq J(\zeta)$, the equality holding if and only if $E=G$, $F=0$ almost everywhere. The first principal theorem is as follows. (I) If ζ_0 minimizes $I(\zeta)$ among all ζ in a convex set of surfaces, then ζ_0 is unique (except for an additive constant). In particular, there is a unique vector minimizing I with given boundary values. A vector ζ (perhaps not continuous on the boundary) which is minimizing on each subregion of G is called an I-surface; if also $E=G$, $F=0$ almost everywhere, it is called an extremal surface. The writer next demonstrates an isoperimetric inequality and various continuity and compactness theorems for I-surfaces and extremal surfaces. The principal theorem is as follows. If Γ bounds two extremal surfaces which are proper relative minima, it must bound an unstable extremal surface. To prove this, the writer concludes from theorem (I) that the space $\mathfrak{A}$ of vectors bounded by a given curve is retractable into the set of I-surfaces in $\mathfrak{A}$. Instead of proving reducibility and applying the Morse theory, the writer then proves the theorem directly for a certain class of curves by considering certain approximating spaces in which the theorem is easily proved and then passing to the limit in a certain way. The result is then extended to hold for any rectifiable boundary curve. *C. B. Morrey, Jr.*

Dutka, Jacques. Transversality in higher space. J. Math. Phys. Mass. Inst. Tech. 23, 126–133 (1944). [MF 11139]

In the space of the variables (x, y, z, v) consider an arbitrary nonsingular analytic one-to-one correspondence T which associates with each hypersurface element

$$(x, y, z, v, \partial v/\partial x, \partial v/\partial y, \partial v/\partial z)$$

at an arbitrary point $P:(x, y, z, v)$ a direction (x', y', z') at P. Generalizing a theorem of Kasner, the author proves that a necessary and sufficient condition that T be the transversality relation associated with some calculus of variations integral

$$\int G(x, y, z, v, y', z', v')dx$$

is that a certain correlation induced between lines and planes through P in the tangent 3-flat to the hypersurface element

$$(x, y, z, v, \partial v/\partial x, \partial v/\partial y, \partial v/\partial z)$$

at P be a polarity. This condition is also equivalent to transversality with respect to

$$\iiint F(x, y, z, v, \partial v/\partial x, \partial v/\partial y, \partial v/\partial z)dxdydz$$

and to a certain correspondence set up by the infinitesimal contact transformation defined by a characteristic function

$$W(x, y, z, v, \partial v/\partial x, \partial v/\partial y, \partial v/\partial z).$$

The results hold also in n-space. *S. B. Myers.*

Functional Analysis

Sobczyk, A. and Hammer, P. C. A decomposition of additive set functions. Duke Math. J. 11, 839–846 (1944). [MF 11585]

Let $f(X)$ be a nonnegative additive set function. Let P denote a finite decomposition of the space M into mutually disjoint sets $X_1, \cdots, X_n$ for each of which $f(X_i)$ is defined. The norm $|P(f)|$ of P for f is the max $(f(X_i), i=1, \cdots, n)$. Then f is called continuous if $\inf_P |P(f)| = 0$. The main result of the present paper is that f can be expressed as the sum $f_0 + \sum f_i$, at most denumerable, of additive functions such that f_0 is continuous and the f_i two-valued. Of course, f_0 may be zero and there may be no $f_1, f_2, \cdots$ present. *F. J. Murray* (New York, N. Y.).

Sobczyk, A. and Hammer, P. C. The ranges of additive set functions. Duke Math. J. 11, 847–851 (1944). [MF 11586]

The notation is the same as that of the preceding review. The authors prove the following. (1) If $f(X)$ is nonnegative

then the range is either finite or a perfect set. (2) There exists an additive $f(X)$ whose range is denumerable. This result is apparently paradoxical, since $f(X)$ must be the difference of nonnegative additive functions for which (1) holds. The function $f(X)$ is constructed under the assumption of well-ordering. *F. J. Murray* (New York, N. Y.).

Wassilkoff, D. Orderings of abstract sets and linear systems. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 7, 203–236 (1943). (Russian. English summary) [MF 10832]

This paper contains a detailed exposition of material previously given without proof [C. R. (Doklady) Acad. Sci. URSS (N.S.) **39**, 167–169 (1943); these Rev. **5**, 186]. The adjoint orderings mentioned at the end of the review of the earlier note arise as follows. Following the notation of that review, let E be a linear system ordered by Ω, and let F be the set of linear functionals defined on E. Now F is ordered by $\bar{\Omega}=\varphi(\Omega)$ in the following manner: $f_1<f_2[\bar{\Omega}]$ if $f_1(x)\leqq f_2(x)$ for all $x\geqq 0[\Omega]$ and if there is at least one x_0 for which $f_1(x_0)<f_2(x_0)$. If an ordering $\bar{\Omega}$ has been defined in F, it in turn defines an adjoint ordering $\Omega=\epsilon(\bar{\Omega})$ in E: $x_1<x_2[\epsilon(\bar{\Omega})]$ if $f(x_1)\leqq f(x_2)$ for all $f\geqq 0[\bar{\Omega}]$ and if $f_0(x_1)<f_0(x_2)$ for at least one $f_0>0$. The following theorem is proved. In order that Ω be adjoint to an ordering $\bar{\Omega}$ it is necessary and sufficient that Ω satisfy axioms R and H^*; if this condition is fulfilled, then (1) $\Omega=\epsilon(\varphi(\Omega))$; (2) Ω satisfies R and H if and only if the equality $f(x)=0$, when valid for all $f\geqq 0[\bar{\Omega}]$, implies $x=0$. A similar theorem, dual to this one, is also proved. *J. V. Wehausen.*

Vulich, B. Sur les opérations linéaires multiplicatives. C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 142–144 (1943). [MF 11058]

The author considers two linear partially ordered topological spaces X and Y satisfying the five axioms of Kantorovitch [Rec. Math. [Mat. Sbornik] N.S. **2(44)**, 121–165 (1937)] and each having a unit in the sense that, for every $x>0$, inf $(x, 1)>0$. In a previous note [same C. R. (N.S.) **26**, 850–854 (1940); these Rev. **2**, 221] the author defined for some elements the notion of the product of two of them. In the present paper he defines a multiplicative operator U on X to Y as one for which $U(x_1)U(x_2)=U(x_1x_2)$ provided $x_1\cdot x_2$ and $U(x_1)\cdot U(x_2)$ exist. He states that an operator U which is continuous in the topological sense ($H_t{}'$ in Kantorovitch's notation) is multiplicative if and only if (a) $x\geqq 0$ implies $U(x)\geqq 0$; (b) if e is a quasi-unit in X, $U(e)$ is a quasi-unit in Y (that is, inf $(e, 1-e)=0$); (c) if e_1, e_2 are quasi-units in X and if inf $(e_1, e_2)=0$, then inf $(U(e_1), U(e_2))=0$. For linear multiplicative operators $U(x^{-1})=U(x)^{-1}$ provided x^{-1} exists. Moreover, the point-wise limit in the topological sense of a sequence U_n of such operators is again multiplicative. A projective operator is defined by Freudenthal as an operator $P_e(x)=\sup_n \inf (ne, x)$, where e is a quasi-unit and $x\geqq 0$. In the note quoted above the author showed that $P_e(x)=xe$ and he now states that a linear multiplicative operator U is projective if and only if $U(x)\leqq x$ for every $x\geqq 0$. He says that a space satisfies the condition (α) of Pinsker if each set of quasi-units contains a finite or denumerable subset with the same supremum. A space is continuous if no quasi-unit $e>0$ is representable as the sum of two positive quasi-units. In the contrary case the space is discrete. In a discrete space there is a nonzero linear multiplicative functional whereas no such functional exists in a continuous space satisfying the condition (α) of Pinsker. There is a linear multiplicative operator which transforms a discrete space into an arbitrary linear partially ordered space containing a unit but there is no such operator which transforms a continuous space satisfying the condition (α) into a discrete space. In the last section of the paper the author defines the notion of a measure function ψ and obtains a representation as a Radon integral for an arbitrary linear multiplicative operator on X_ψ to X, where X_ψ is the set of all x in X which are summable with respect to ψ.

H. H. Goldstine (Philadelphia, Pa.).

Vulich, B. Sur la représentation analytique d'opérations linéaires multiplicatives. C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 187–190 (1943). [MF 11063]

The author considers a number of examples of partially ordered spaces and gives representations for arbitrary linear multiplication operators defined on such spaces. In particular he considers the spaces L^p ($p\geqq 1$), M, S, the space of all real measurable functions that are finite almost everywhere on $[a, b]$, l^p, m, s and c_0. *H. H. Goldstine.*

Hamburger, Hans Ludwig. Hermitian transformations of deficiency-index (1, 1), Jacobi matrices and undetermined moment problems. Amer. J. Math. **66**, 489–522 (1944). [MF 11392]

This paper is based on two previous ones [Quart. J. Math., Oxford Ser. **13**, 117–128 (1942); Ann. of Math. (2) **45**, 59–99 (1944); these Rev. **5**, 40, 188]. Here the author determines, when H is a given closed Hermitian prime transformation of deficiency index (1, 1), necessary and sufficient conditions for the existence of a complete orthonormal set of elements u_1, u_2, $\cdots$, such that $(Hu_i, u_j)=a_{ij}=0$, $|i-j|\geqq 2$, thus finding necessary and sufficient conditions for H to be carried into a Jacobi matrix of deficiency index (1, 1). As a consequence of a result of M. H. Stone [Linear Transformations in Hilbert Space and their Applications to Analysis, Amer. Math. Soc. Colloquium Publ., v. 15, New York, 1932, p. 585, theorem 10.41], the author only needs to consider the case in which the self-adjoint extensions of H have spectra consisting only of an infinite number of isolated points. Accordingly he first shows that, if H is a closed Hermitian prime transformation of deficiency index (1, 1) such that the spectrum of one of its self-adjoint extensions consists only of an infinite number of isolated points, there exist four integral functions of x, satisfying certain conditions, which determine the resolvents $R_x{}^t$ of all self-adjoint extensions $H_x{}^t=H^t-xI$ of H. (By the theory previously developed by the author, an mth order matrix $C(x)$ is defined by any self-adjoint extension of a closed Hermitian prime transformation of deficiency index (m, m). If $m=1$, $C(x)=C(x;t)$, where $-\infty<t\leqq\infty$ and H^t is that self-adjoint extension of H which determines $C(x;t)$.) The necessary and sufficient conditions mentioned above are then expressed as conditions on these four integral functions of x.

Since every Jacobi matrix of deficiency index (1, 1) is associated with a sequence defining an undetermined moment problem, the author concludes by proving the following theorem. Let $\mathfrak{A}$ be the class of all integral functions $q(x)$ of finite order, real for real x, whose roots λ_n are all real and

simple, and which satisfy the two conditions

$$\frac{1}{q(x)}=\sum_{\alpha=1}^{\infty}\frac{1}{q'(\lambda_\alpha)(x-\lambda_\alpha)},\quad \sum_{\alpha=1}^{\infty}\frac{\lambda_\alpha^k}{q'(\lambda_\alpha)}<\infty,\quad k=0,1,2,\cdots.$$

Then we find all sequences $\{c_k\}$ defining an undetermined moment problem by associating with any $q(x)$ of class $\mathfrak{A}$ a sequence $\{\mu_\alpha\}$, $\mu_\alpha>0$, such that

$$\sum_{\alpha=1}^{\infty}\mu_\alpha=1,\quad \sum_{\alpha=1}^{\infty}\mu_\alpha\lambda_\alpha^{2k}<\infty,\qquad k=1,2,\cdots,$$

$$\sum_{\alpha=1}^{\infty}\frac{1}{\mu_\alpha(q'(\lambda_\alpha))^2}=\infty,\quad \sum_{\alpha=1}^{\infty}\frac{1}{\mu_\alpha\lambda_\alpha^2(q'(\lambda_\alpha))^2}<\infty.$$

If we put $c_k=\sum_{\alpha=1}^{\infty}\mu_\alpha\lambda_\alpha^k$, $k=0,1,2,\cdots$, the sequence $\{c_k\}$ defines an undetermined moment problem and the solution thus obtained is a maximal distribution of masses.

J. Williamson (Flushing, N. Y.).

Krein, M. On Hermitian operators whose deficiency indices are 1. C. R. (Doklady) Acad. Sci. URSS (N.S.) **43**, 323–326 (1944). [MF 11620]

The results announced in this paper concern a closed Hermitian operator on a Hilbert space $\mathfrak{H}$ whose domain $\mathfrak{D}_A$ is dense in $\mathfrak{H}$ and whose deficiency indices are all equal to 1; that is, the orthogonal complements relative to $\mathfrak{H}$ of the classes $\mathfrak{M}_z$ of elements expressible in the form $Af-zf$, f in $\mathfrak{D}_A$, are one-dimensional for Jz (imaginary part of z)$\neq 0$. If u does not belong to $\mathfrak{M}_z$ for at least one z in each half plane, the set $\mathfrak{E}_u$ of points α ($J\alpha\neq 0$) such that u belongs to $\mathfrak{M}_\alpha$ has limiting points only on the real axis, and

$$\prod_{\alpha\in\mathfrak{E}_u}|\alpha+z|/|\alpha-z|$$

is convergent. The element u sets up a one to one correspondence between the elements f of $\mathfrak{H}$ and functions $f_u(z)$ whose only singularities are poles belonging to $\mathfrak{E}_u$, via the condition that $f-f_u(z)u$ belongs to $\mathfrak{M}_z$. In particular, for $f=u$, $f_u(z)=1$, and if $f=Ag$, then $g_u(z)=zf_u(z)$. The operator A defines a class of spectral functions E_t (a spectral function being a one parameter set of self-adjoint operators on $\mathfrak{H}$ such that (E_tf,f) is monotonic nondecreasing in t, E_t is left-continuous in t, $E_{-\infty}f=0$ and $E_\infty f=f$, and $(Af,Af)=\int_{-\infty}^{\infty}t^2d(E_tf,f)$, $Af=\int_{-\infty}^{\infty}tdE_tf$ for all f of A). The problem of determining the class of functions $w_\sigma(z)=\int_{-\infty}^{\infty}d\sigma(t)/(t-z)$ corresponding to the class V_u of monotonic nondecreasing functions $\sigma(t)=(E_tu,u)$, where E_t ranges over the spectral functions of A, is solved in the theorem that a necessary and sufficient condition that $w_\sigma(z)$ correspond to a σ of V_u is that $w_\sigma(z)$ lie in a certain circle $C(z)$ lying in the half plane determined by z, for each z, $Jz\neq 0$.

T. H. Hildebrandt (Ann Arbor, Mich.).

Livshitz, M. S. On an application of the theory of Hermitian operators to the generalized problem of moments. C. R. (Doklady) Acad. Sci. URSS (N.S.) **44**, 3–7 (1944). [MF 11604]

If G is a linear set of complex valued functions on $-\infty<t<\infty$ and P is a linear functional on $G\bar{G}$ (the linear extension of the class of functions $f(t)\bar{g}(t)$) which is positive in the sense $P(f\bar{f})\geqq 0$ for all f of G, then $P(f\bar{g})$ takes the form $\int_{-\infty}^{\infty}f(t)g(t)d\sigma(t)$ if (a) $f(t)\equiv 1$ is in G, (b) the functions g of G for which tg belongs to G are dense in G in the sense of the P-norm $[P(f\bar{f})]^{\frac{1}{2}}$, the form being valid for those continuous functions of G for which $(f(t)-f(x))/(t-x)$ considered as a function of t belongs to G for $-\infty<x<\infty$. The proof is made to depend on the theorem: if A is a Hermitian operator on a Hilbert space $\mathfrak{H}$ whose domain $\mathfrak{D}_A$ is dense in $\mathfrak{H}$, and if f and u of $\mathfrak{H}$ are such that there exists a continuous number function $f_u(x)$ such that $f-f_u(x)u=Ag_x-xg_x$ on $-\infty<x<\infty$, then $f=\int_{-\infty}^{\infty}f_u(t)dE_tu$, where E_t is a spectral function belonging to A. To apply the theorem, G is enlarged to a Hilbert space via the functional $P(f\bar{g})$, $u\equiv 1$, $Ag=tg$ and $\sigma(t)=P[E_t1,1]$. Application is made to a theorem on moments, and to positive definite Hermitian functions [see, for example, Bochner, Math. Ann. **108**, 378–410 (1933)]. *T. H. Hildebrandt.*

Ambrose, Warren. Spectral resolution of groups of unitary operators. Duke Math. J. **11**, 589–595 (1944). [MF 11092]

If x is the point of a locally compact Abelian group G and ξ is the point of its character group G^* and if $[x,\xi]$ is the corresponding character function generalizing $e^{i\xi x}$, then a continuous positive definite function $f(x)$ on G can be represented as an integral

$$f(x)=\int[x,\xi]d_\xi\mu,\qquad d\mu\geqq 0,$$

where μ is a set function on G^*. Hence the author derives as a generalization of Stone's theorem a spectral form

$$U_x=\int[x,\xi]d_\xi E$$

in which U_x represents a group of unitary transformations on some Hilbert space and $d_\xi E$ is an associated suitable resolution of the identity. This theorem is also contained in a paper by M. Neumark [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **7**, 237–244 (1943); these Rev. **5**, 272]. *S. Bochner* (Princeton, N. J.).

Halmos, Paul R. In general a measure preserving transformation is mixing. Ann. of Math. (2) **45**, 786–792 (1944). [MF 11382]

A measure preserving transformation T on a (normed) space E is mixing (in the weak sense of E. Hopf) if there exists a set I_0 of positive integers of density zero such that, for any two measurable sets a and b of E,

$$\lim_{n\bar{\epsilon}I_0}|T^na\cdot b|=|a||b|,$$

where $|a|$ denotes the measure of a. This paper solves one of the outstanding mixing problems by showing that the nonmixing transformations form a set of the first category in the transformation space with one of its natural topologies.

The proof is carried through for the nonrestrictive case when E is the unit interval I. If G denotes the group of all measure preserving transformations of I onto itself, $S\epsilon G$, a is a measurable set in I, and $\epsilon>0$, let

$$N(S)=N(S;a,\epsilon)=\{T:|Sa-Ta|<\epsilon\},$$

where $a-b$ stands for $ab'\cup a'b$, and a' is the complement of a. By the requirement that the sets $N(S)$ be a subbase for open sets a unique (neighborhood) topology is defined in G. A measure preserving transformation T is (almost everywhere) nonperiodic if the set of periodic points of T is a set of measure zero.

If M denotes the set of all mixing transformations in G, it is sufficient to prove that M is everywhere dense in G

and is a G_δ. To prove that M is everywhere dense in G it is sufficient to show that: (i) M contains a nonperiodic transformation, (ii) M is self-conjugate, (iii) the conjugate class of any nonperiodic measure preserving transformation of G is everywhere dense in G. The first two of these properties are almost self-evident. The proof of the third is difficult. Extensive use is made of a previous paper by the author [Trans. Amer. Math. Soc. **55**, 1–18 (1944); these Rev. **5**, 189] and of the density properties of the dyadic permutations. The proof that M is a G_δ requires use of the fact that a measure preserving transformation may be regarded as a unitary operator in Hilbert space and of various results which have been obtained concerning this characterization.

G. A. Hedlund (Charlottesville, Va.).

Cameron, R. H. and Martin, W. T. The Wiener measure of Hilbert neighborhoods in the space of real continuous functions. J. Math. Phys. Mass. Inst. Tech. **23**, 195–209 (1944). [MF 11414]

Let C be the space of continuous functions $x(t)$ vanishing when $t=0$. The measure of C-sets used by the authors is the usual measure defined in the study of Brownian movements, first defined rigorously by Wiener. The authors find the measure of the set of functions satisfying the inequality $\int_0^1 x(t)^2dt<R^2$, and evaluate related integrals. Their work could be somewhat abbreviated by the use of various probability theorems such as Lévy's on the evaluation of a distribution function in terms of its characteristic function.

J. L. Doob (Washington, D. C.).

NUMERICAL AND GRAPHICAL METHODS

Bateman, Harry and Archibald, Raymond Clare. A guide to tables of Bessel functions. Mathematical Tables and Other Aids to Computation **1**, 205–308 (1944). [MF 11276]

This issue of the journal on computations published by the National Research Council [cf. the bibliographical note, these Rev. **4**, 152] is a special number. It contains a list of the tremendous literature on tables and graphs concerning the various Bessel functions. The great importance of such a collection is obvious.

G. Szegö.

Abramowitz, Milton. Zeros of certain Bessel functions of fractional order. Mathematical Tables and Other Aids to Computation **1**, 353–354 (1945). [MF 11806]

The tables contain the zeros of $J_\nu(x)$ for $x\leqq 25$, where $\nu=\pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{1}{3}, \pm\frac{2}{3}$. These zeros were obtained by inverse interpolation in a thirteen-place manuscript of these functions. The accuracy to 10 decimal places is guaranteed, and the two additional places have a high probability of being correct.

Extract from the paper.

Strscheletzky, M. Annähernde Lösung des Integrals $J_m(z)=\frac{1}{\pi}\int_{\omega_1}^{\omega_2} e^{iz\cos\omega}\cdot e^{im(\omega-\pi/2)}\cdot d\omega$. Z. Angew. Math. Mech. **23**, 295–296 (1943). [MF 11741]

For computational purposes, the integral is expanded into a series.

Salzer, Herbert E. Table of coefficients for differences in terms of the derivatives. J. Math. Phys. Mass. Inst. Tech. **23**, 210–212 (1944). [MF 11415]

The forward differences of the equally spaced values of a function $f(x)$ with tabular interval h can be expressed by Markov's formula in the form

$$\Delta_h^m f(a)=\sum_{s=m}^{n-1} B_{m,s}h^sD^sf(a)+\text{remainder}.$$

The table gives the exact fractional values of the coefficients $B_{m,s}$ for $m=1, 2, \cdots, 20$ and $s=m, \cdots, 20$.

W. E. Milne (Corvallis, Ore.).

Salzer, Herbert E. Table of coefficients for inverse interpolation with advancing differences. J. Math. Phys. Mass. Inst. Tech. **23**, 75–102 (1944). [MF 10629]

If a function $f(x)$ is tabulated for a series of equidistant arguments $x_i=x_0+ih$ and if $f(x)$ is monotonic in (x_0, x_0+h), the problem of inverse interpolation is to find the root $x=x_0+ph$ of $y=f(x)$. H. T. Davis [Tables of the Higher Mathematical Functions, vol. I, The Principia Press, Bloomington, Ind., 1933, pp. 80–81] has derived the formula

$$(1)\quad p=m+\frac{m(1-m)}{2}\frac{\Delta^2}{\Delta}+\frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta}+$$
$$+\frac{m(m-1)(2m-1)}{4}\left(\frac{\Delta^2}{\Delta}\right)^2-\frac{m(m-1)(m-2)(m-3)}{24}\frac{\Delta^4}{\Delta}+\cdots,$$

where $m=(y-y_0)/\Delta$. The author provides tables of the coefficients of all terms up to the eighth order, where the order of the term $(\Delta^i/\Delta)^r(\Delta^j/\Delta)^s\cdots$ is defined by $[ir+js+\cdots]$.

The argument m is at interval .001 for coefficients of the fourth and fifth orders, at .01 for the sixth order, and at 0.1 for the seventh and eighth orders, with coefficients tabulated to 10 decimals and accurate to within $\frac{1}{2}$ or 1 unit in the last decimal. For coefficients of the second and third order terms the user is referred to the "Table of Lagrangian Interpolation Coefficients" published by the Mathematical Tables Project [Columbia University Press, New York, 1944; these Rev. **5**, 244]. The convergence in (1) is of course slower than in the corresponding expansion in terms of central differences for which the author has previously provided tables; the present tables are intended in particular for inverse interpolation at the beginning (or end) of a table. Coefficients are arranged in separate tables so that the user has to look up the argument several times. The number of decimals should have been cut down for the higher order coefficients in order to accommodate all required coefficients in one table.

In practical computation one should avoid the use of the eleven terms of the seventh and eighth order, and instead subtabulate the original table of $f(x)$ with a decidedly simpler method of direct interpolation.

H. O. Hartley.

Neuschuler, L. Sur un nouveau type de tableaux de fonctions à plusieurs variables. C. R. (Doklady) Acad. Sci. URSS (N.S.) **43**, 142–146 (1944). [MF 11614]

Aparo, Enzo. Di alcune avvertenze sulla risoluzione numerica delle equazioni algebriche. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **4**, 125–147 (1943). [MF 11520]

This is an expository article which presents one particular method for finding the upper and lower limits on the roots of an equation $x^n+a_1x^{n-1}+\cdots+a_n=0$. If the a_j are real with a_p as the first negative coefficient and if, for an arbitrary positive number λ,

$$A(\lambda)=\max\,(-a_{p+k}\lambda^{-k})\quad\text{for } k=0, 1, 2, \cdots, n-p,$$

then $\lambda+(A(\lambda))^{1/p}$ is an upper limit for the positive roots of the given equation. This theorem, for which the author refers to Picone, reduces to a well-known theorem of Lagrange when we set $\lambda=1$. In the usual manner, the theorem is extended to give the lower limits for the positive roots and both limits for the negative roots. If the coefficients are complex, a ring $R_1\leqq|x|\leqq R_2$ in which all the roots lie may be found by applying the theorem to the equation $x^n-|a_1|x^{n-1}-\cdots-|a_n|=0$. Numerical examples and applications of these theorems are also included.

M. Marden (Milwaukee, Wis.).

Richmond, H. W. On certain formulae for numerical approximation. J. London Math. Soc. 19, 31–38 (1944). [MF 11318]

The author discusses a generalization of Newton's method for approximating a solution of an equation. Whereas in Newton's method the approximations are improved by intersecting a tangent with the x-axis, the author uses an osculating hyperbola for this purpose. It is shown that the error of any approximation is very nearly proportional to the cube of the preceding error if an osculating hyperbola is used, but proportional to the square of the preceding error if the tangent is used. The conditions for the convergence of the process are not discussed. Similar formulae for the extraction of $x^{1/n}$ were given by G. W. Ward [Math. Gaz. 17, 52–53, 127 (1933)]. *E. Lukacs.*

Mikeladze, Š. E. On formulas for mechanical cubatures, containing partial derivatives of the integrand. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 4, 297–300 (1943). (Russian. Georgian summary) [MF 11703]

The author evaluates a double integral over a region D by dividing D into squares and applying the formula

$$\int_{a-h}^{a+h}dx\int_{b-h}^{b+h}\psi(x,y)dy\approx(4h^2/15)[19\psi(a,b)-\psi(a+h,b)-\psi(a-h,b)-\psi(a,b+h)-\psi(a,b-h)]+(14/15)h^4\Delta\psi(a,b)+(h^6/18)\Delta\Delta\psi(a,b)$$

to a square of side $2h$. *W. E. Milne* (Corvallis, Ore.).

Mikeladze, Š. E. New formulas for the numerical integration of differential equations. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 4, 215–218 (1943). (Russian. Georgian summary) [MF 11700]

These "new" formulas consist of Newton's three-eighths rule

$$y_3-y_0=(3h/8)(y_3'+3y_2'+3y_1'+y_0'),$$

the formula

$$y_2-y_1=(h/24)(-y_3'+13y_2'+13y_1'-y_0')$$

and various linear combinations of these two.

W. E. Milne (Corvallis, Ore.).

Saibel, Edward. On the method of collocation. J. Franklin Inst. 238, 107–110 (1944). [MF 10868]

The author reviews approximate methods of finding the characteristic values λ of a linear differential equation of nth order

$$(1)\qquad L\{y\}+\lambda m(x)y(x)=0$$

with homogeneous boundary conditions. Modifying a method given in Biezeno and Grammel [Technische Dynamik, Springer, Berlin, 1939; Edwards Bros., Ann Arbor, Mich., 1944], he considers the associated integral equation

$$(2)\qquad y(x)=\int_0^l m(s)G(x,s)y(s)ds,$$

where $G(x,s)$ is Green's function corresponding to (1). Putting now $y(x)=\sum b_rY_r(x)$, and substituting in (2), this equation is to be satisfied exactly at n selected values of x. This yields a determinantal equation for λ. The choice of the $Y_r(x)$ is not discussed but presumably they are to be taken from a complete orthogonal set. No proof is given that the roots λ will approximate to the first n characteristic values of (1). As an example the equation $y^{iv}+c\lambda y=0$, $0\leqq x\leqq l$; $y(0)=y'(0)$; $y''(l)=y'''(l)=0$, is dealt with and the lowest natural frequency of vibration of a uniform cantilever beam is approximated to by an inspired guess of a single term $y(x)$. *H. O. Hartley* (London).

Moskovitz, David. The numerical solution of Laplace's and Poisson's equation. Quart. Appl. Math. 2, 148–163 (1944). [MF 10808]

The author deals with an approximate method of solving the first boundary value problem for the equations $\Delta u=0$ and $\Delta u=F(x,y)$ for a rectangle $0\leqq x\leqq nh$, $0\leqq y\leqq mh$. The procedure (sometimes referred to as Liebmann's) is to cover the rectangle by a grid of lattice points $x=ih$, $y=jh$, $i=1,\cdots,n$; $j=1,\cdots,m$, and to set up the equation

$$(1)\quad 4u(x,y)=u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)[-h^2F(x,y)]$$

at each interior point x, y of the grid. When a boundary point is included in an equation the value of u is prescribed. The values of u in the interior are the unknowns.

The author rewrites (1), in the case $F=0$, as

$$L_Cu_j(i)\equiv Cu_j(i)-u_j(i+1)-u_j(i-1)=u_{j+1}(i)+u_{j-1}(i)+\phi_j(i),$$

where $\phi_j(i)$ depends on the prescribed boundary values, $C=4$ and $u_j(i)=u(ih,jh)$. A formal solution is reached giving the $u_j(i)$ in terms of linear aggregates of the inverse operators L_C^{-1} applied to the boundary values. For $m=2$, 3 and 4 the operators are (in certain cases) evaluated numerically and tables are prepared with the help of which the $u_j(i)$ may be obtained from the boundary values. This numerical process appears to be equivalent to the "back solution" in a special Gauss-Doolittle elimination method applied to the original set of equations (1).

The Poisson equation $\Delta u=F(x,y)$ is treated in a similar manner. Certain difficulties have to be overcome when the sides of the rectangle are not commensurable. The generalization to nonrectangular regions is announced. The representation of the $u_j(i)$ in terms of the operator $L_Cu_j(i)$ has certain numerical advantages over the ordinary determinantal representation of the solution. No comparison is made with the formal representation of the exact solution with the help of Green's function for the rectangle, which appears to be simpler. *H. O. Hartley* (London).

Monfraix, Paul. Théorie générale des intégrateurs à roulette coupante. C. R. Acad. Sci. Paris 216, 865–867 (1943). [MF 10658]

The author develops a general theory of integrators of the planimeter type. The relations between the coordinate system of the movable part and that of the fixed basic plane are derived for nonslip movements of the wheel along a curve in the basic plane. It is shown how by various restrictions of this movement the coordinates in the movable system can be made to satisfy a number of special differen-

tial equations of the first order. The standard properties of planimeters are also derived as special cases. The mechanical realization of the various restrictions of movements is not discussed. *H. O. Hartley* (London).

Fischer, Ernst. Das Zinsfussproblem der Lebensversicherungsrechnung als Interpolationsaufgabe. Mitt. Verein. Schweiz. Versich.-Math. **42**, 205–307 (1942). [MF 11456]

Dasen, E. Note sur l'approximation du taux effectif des emprunts par obligations amortissables par le système de l'annuité constante. Mitt. Verein. Schweiz. Versich.-Math. **41**, 201–204 (1941). [MF 11452]

Stauber, Kurt. Beiträge zur Theorie der Kompakttafel. Mitt. Verein. Schweiz. Versich.-Math. **42**, 97–146 (1942). [MF 11450]

Schuler, Werner Peter. Ein Verfahren zum Einbezug der säkularen Sterblichkeitsabnahme in die versicherungstechnischen Berechnungen. Mitt. Verein. Schweiz. Versich.-Math. **44**, 107–149 (1944). [MF 11460]

Tortorici, Paolo. Sopra un nuovo metodo per la determinazione del tasso d'investimento in un prestito rimborsabile in unica volta. Ist. Naz. Appl. Calcolo (2) no. **159**, 13 pp. (1944). [MF 11765]

Rymer, T. B. and Butler, C. C. An electrical circuit for harmonic analysis and other calculations. Philos. Mag. (7) **35**, 606–616 (1944). [MF 11800]

A potentiometer circuit is described in which two resistors are adjusted to represent p and q. A third resistor can then be set to have a value in proportion to the product pq. Using a resistance network for addition, component products can be combined to give the average value of $p_n q_n$ with a single setting of the output resistor. These operations form the basis for manipulating a group of 18 ordinates in order to obtain Fourier coefficients. Provision is made for changing the sensitivity of the output circuit so that precision can be maintained in measuring the higher order coefficients. Since the basic operation uses direct current, it is possible to superpose an alternating-current system which prevents gross error by checking the continuity of the contact system used. This equipment does not give the Fourier coefficients directly, but provides a set of coefficients which are related to the Fourier coefficients. The authors give one example of 36 machine coefficients obtained in two hours' time. To this must be added the time required to convert to Fourier coefficients. It should be possible at least to equal this speed, and perhaps to exceed it, by conventional schedule computations.
S. H. Caldwell (Cambridge, Mass.).

Thomas, George B. Preparation of punched-card tables of logarithms. Rev. Sci. Instruments **15**, 350 (1944). [MF 11591]

King, Gilbert W. Punched-card tables of the exponential function. Rev. Sci. Instruments **15**, 349–350 (1944). [MF 11590]

Frame, J. S. Machines for solving algebraic equations. Mathematical Tables and Other Aids to Computation **1**, 337–353 (1945). [MF 11805]

MECHANICS

Alt, H. Die Kardanlagen von Getriebegliedern und die Krümmung der Polkurven. Ing.-Arch. **14**, 319–331 (1944). [MF 11441]

A plane in plane motion is said to be in a Cardan position if the instantaneous motion (velocities and accelerations) can be generated by letting a circle roll on another circle of twice its radius. Various properties of such positions are analyzed. The results are applied to the cranks of a four bar linkage. However, not every linkage passes through Cardan positions. *W. Feller* (Providence, R. I.).

Hackmüller, E. Die Ermittlung von Koppeltrieben aus zwei Kurbel-Schubkurventrieben. Ing.-Arch. **14**, 141–154 (1943). [MF 11434]

Consider two rigid links AB and BC which can freely turn about the joint B. The endpoint A of the first (the driven) link is fixed, while the endpoint C moves along a prescribed path Γ. The ordinary four bar linkage is a special case with Γ a circle; BC is then the connecting rod. The author shows, by examples, how the use of complex numbers simplifies the necessary computations in determining the motion of the linkage. *W. Feller* (Providence, R. I.).

Grammel, R. Über Schwingungsketten. I. Ing.-Arch. **14**, 213–232 (1943). [MF 11437]

This paper deals with linear dynamical systems, of the kind exemplified by a finite number of massive particles, constrained to move on a more or less complicated curve, and being interconnected by springs in a more or less complicated manner. The systems are classified according to the topological properties of the curve, and according to the connections between the particles. Various theorems (some of which are not new) are given, relating to the natural frequencies and modes of vibration of the systems.
L. A. MacColl (New York, N. Y.).

Couffignal, Louis. Sur les conditions de stabilité des systèmes oscillants. C. R. Acad. Sci. Paris **217**, 594–596 (1943). [MF 11675]

Let the determinantal equation of a linear dynamical system be written in the form $\sum a_i p^{2n-i}=0$. Routh gave a necessary and sufficient condition for the stability of the system in the form of a set of inequalities applying to certain functions of the a's. The present author gives an alternative condition for stability of the same kind. In simple cases his condition admits of geometrical interpretations, which are discussed briefly. *L. A. MacColl.*

Minorsky, N. On mechanical self-excited oscillations. Proc. Nat. Acad. Sci. U. S. A. **30**, 308–314 (1944). [MF 11358]

As an analogue to the antirolling stabilization of a ship by means of activated tanks, the author considers a rigid physical pendulum P, to which is secured a liquid pendulum P'. The latter consists of a U tube joining two tanks and is filled with liquid. In the U tube is inserted a variable pitch pump run at constant speed. Activation is controlled by varying the pitch angle α. The pitch angle α is itself controlled electronically by the angular motion of P. The optimum condition of control is obtained by having $\alpha=\lambda\dot{\theta}$,

where λ is an adjustable constant and $\ddot{\theta}$ is the angular acceleration of P. If λ exceeds a certain value λ_0, marked self-excitation occurs in P' even with P almost at rest.

The phenomenon is explained by the nonlinear characteristic of the pump, $M=M_1\alpha-M_2\alpha^3$, where M is the moment of the force of the pump acting on P' and M_1 and M_2 are constants. Introducing the angular coordinate ϕ in P' and considering the case where P is almost at rest leads to the equation

$$\ddot{\phi}+g(\lambda)\dot{\phi}+m\dot{\phi}^3+\omega_0^2\phi=0, \qquad m>0,$$

where $g(\lambda)$ is a decreasing function of λ. The differential equation is of the Rayleigh type. It has an oscillatory solution as its only stable stationary solution if $g(\lambda)<0$, and has $\phi=0$ as its only stable stationary solution if $g(\lambda)>0$. Thus λ_0 is the root of the equation $g(\lambda)=0$ and, if $\lambda>\lambda_0$, the system is self-excited. *N. Levinson* (Cambridge, Mass.).

Pugachev, V. S. Approximate method of solving the nonlinear problem for the motion of a rotating projectile. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 313–324 (1943). (Russian. English summary) [MF 11234]

The author sets up the differential equations for the rotary motion of an artillery projectile under quite general assumptions, taking into consideration the lateral force, the force of Magnus, the turning moment, the moment of Magnus, the polar damping moment and the ~~equational~~ equatorial damping moment. These forces and moments are assumed to be expressible as Fourier series in terms of the angle of yaw δ, with coefficients which are functions of y and v. Using only one or two terms of the Fourier series for the forces and moments, he obtains a set of differential equations sufficient to determine the unknowns, but not integrable in elementary form. The plan of attack is to express the coordinates necessary to describe the flight of the projectile together with its precession and oscillation in terms of parameters which are nonoscillating functions of time. For example, the quantity $z=\sin^2(\delta/2)$ is expressed in the form $z=Z_0+Z_1\cos\psi$, where $\psi=\int w\,dt$ and where Z_0, Z_1 and w are smoothly varying functions of time. These latter quantities are determined by differential equations which can be solved by numerical integration. The paper would have been clearer if the methods had been illustrated by the solution of an actual problem for a typical projectile.

W. E. Milne (Corvallis, Ore.).

Hydrodynamics, Aerodynamics

Thomas, T. Y. On the stability of viscous fluids. Univ. California Publ. Math. (N.S.) **2** [No. 1, Seminar Rep. in Math. (Los Angeles)], 13–43 (1944). [MF 10453]

This paper establishes certain sufficiency conditions for the stability of various flows of an incompressible viscous fluid in a region R with varied boundaries. The author defines stability as follows. Let S denote a stationary fluid motion in R having velocity components $\bar{u}^a$ ($a=1, 2, 3$), and let M denote a class of fluid motions in R. Then S is stable relative to M if $u^a\rightarrow\bar{u}^a$ uniformly over R as $t\rightarrow\infty$, where the u^a denote the velocity components of an arbitrary motion of M. It is assumed that the bounding surface B can be represented locally by $x^a=x^a(u, v)$ with continuous first derivatives, that the matrix $\|x_u^a\ x_v^a\|$ has rank two at each point and that the functions u^a and the pressure p satisfy the following conditions: u^a have derivatives with respect to the space coordinates to the order three inclusive and a continuous first derivative with respect to the time; p is continuous with respect to time and all its first derivatives are continuous with respect to the space coordinates; u^a and their derivatives with respect to the space coordinates are uniformly bounded.

One of the main results is the following. Let a fluid, on which no external forces act, completely fill the space between one or more fixed surfaces forming a boundary B (closed and bounded); then the state of rest of the fluid ($u^a=0$) is stable relative to arbitrary fluid motions (satisfying the above conditions). The rest of the paper is concerned largely with the stability of flows under varied conditions, notably, when the boundaries are in simple rigid motion, for Conette motion relative to plane rotational symmetric motion and for Poiseuille motion in an infinite circular cylinder.

The author also proves that, if R is the Reynolds number and R_e is any positive constant, the condition $R<R_e$ is not sufficient to prove stability in the sense defined above (in particular, the stability of Poiseuille motion relative to axial symmetric motion) in an infinite cylinder on the basis of the equation of continuity and the Navier-Stokes equation when the usual boundary conditions alone are imposed (vanishing of the velocities over the surface of the cylinder).

A. Gelbart (Syracuse, N. Y.).

Valentine, F. A. On the stability of a compressible viscous fluid. Univ. California Publ. Math. (N.S.) **2** [No. 1, Seminar Rep. in Math. (Los Angeles)], 153–159 (1944). [MF 10459]

Under definitions and assumptions similar to those made in the paper reviewed above, the author extends some of its results to the motion of a compressible viscous fluid. He proves that, if the density ρ of the fluid is uniformly bounded with respect to the time, and if the relation between the pressure and the density has the form

$$p=a_0+\sum a_n\rho^{\epsilon_n}, \qquad 0<\epsilon_n\neq 1,$$

where a_n and ϵ_n are positive constants, then the motion $\bar{u}^a(x)=0$, $x\epsilon B$, is stable relative to the set of all admissible motions, B being the boundary of the region at rest in Euclidean space. *A. Gelbart* (Syracuse, N. Y.).

Landau, L. A new exact solution of Navier-Stokes equations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **43**, 286–288 (1944). [MF 11619]

Suppose an axially symmetrical jet discharges from a thin pipe into unbounded space. From the assumption that the components $\Pi_{\theta\theta}$ and $\Pi_{\varphi\varphi}$ of the tensor of momentum flow density are zero, the author shows that all other components of Π_{ik} are zero except Π_{rr}, and deduces that the Navier-Stokes equations are satisfied identically. He then solves the equation $(1/\rho)(\Pi_{\theta\theta}-\Pi_{\varphi\varphi})=0$ to determine the velocity components and other features of the flow. The limiting cases of weak and strong jets are specifically considered.

C. C. Torrance (Washington, D. C.).

Schultz-Grunow, F. Nichtstationäre, kugelsymmetrische Gasbewegung und nichtstationäre Gasströmung in Düsen und Diffusoren. Ing.-Arch. **14**, 21–29 (1943). [MF 11429]

The paper is concerned with an approximate method of treating spherically symmetric, unsteady flows of a compressible fluid. It is pointed out that the equations of motion for plane and spherical waves of finite amplitude are

identical, while the equations of continuity differ only by an extra term appearing in the equation for the spherical wave. The equation of continuity for a plane wave in a channel of variable width possesses a similar additional term. A spherical wave may thus be treated as a plane wave in a channel whose width is made to vary in an appropriate manner. An approximate method of treating spherical waves is obtained by replacing the channel of continuously varying width by one of piecewise constant width.

W. Prager (Providence, R. I.).

Sauer, R. Zur Theorie des nichtstationären ebenen Verdichtungsstosses. Ing.-Arch. **14**, 14–20 (1943). [MF 11428]

The graphical method developed in an earlier paper [Ing.-Arch. **13**, 79–89 (1942); these Rev. **4**, 260] is extended to include the occurrence of "weak" shocks. The change of state across such a shock is considered as practically isotropic and the shock velocity is taken as the arithmetic mean of the velocities of sound at the two sides of the shock.

W. Prager (Providence, R. I.).

Bechert, Karl. Ebene Wellen in idealen Gasen mit Reibung und Wärmeleitung. Ann. Physik (5) **40**, 207–248 (1941). [MF 11195]

By choosing as independent variables the mass $m=\int\rho dx$ per unit area perpendicular to the axis of x, and the time t, the equations for the variable flow of an ideal gas in one direction are expressed in the form

$$v_t-u_m=0,\ [u-\mu(\log v)_m]_t+(CT/v)_m=0,$$

$$(hCT+\tfrac{1}{2}u^2)_t+[CTu/v-\mu u(\log v)_t-\lambda_1 CT_m/v]_m=0,$$

where u is the velocity, v the specific volume, T the absolute temperature, C is the ratio R/M of the universal gas constant R to the molecular weight and hC is the specific heat at constant volume. In these equations suffixes denote partial derivatives with respect to the variables m and t. The coefficient μ represents the viscosity and $C\lambda_1$ the thermal conductivity. By the substitutions $pv=CT$, $C_v=hC=C/(k-1)$, $\lambda_1=h\lambda_2$, $p=-V_t$, $v=S_m$, $u=S_t$, $S=K_m$, $\mu\log v+V=K_t$, $u-\mu(\log v)_m=V_m$, a partial differential equation

$$\mu K_{mmtt}-kK_{mmt}K_{tt}-K_{mm}K_{ttt}$$
$$-\lambda_2[(K_{mmt}\mu-K_{mm}K_{tt})_m/K_{mm}]_m=0$$

is obtained for the determination of K, but instead of this equation use may be made of the equations

$$P=p-\mu(v_t/v),\quad CT=pv=Pv+\mu v_t,\quad P_{mm}+v_{tt}=0,$$

$$kPv_t+P_tv+\mu v_{tt}=\lambda_2(P_m+(Pv_m+\mu v_{mt})/v)_m=0.$$

From P and v the quantity K can be found and then the other quantities u, p, etc. can be calculated.

Some special solutions are considered and it is found that the differential equations give some solutions which are not physically realizable or not quite realizable. The reason for this may be that the gas is supposed to be ideal. Some interesting remarks are made about the condition under which shock occurs. As in the work of Bjerknes the function $f(m)$ in the adiabatic relation (for $\lambda_1=\mu=0$) is not assumed to be constant and then shock seems to occur when the state quantities attain limiting values.

H. Bateman.

Morton, W. B. The paths of the particles in a vortex street. Proc. Roy. Irish Acad. Sect. A. **49**, 289–292 (1944). [MF 11474]

The author calculates the stream lines and the paths of particles in a von Kármán vortex street, relative both to an observer fixed in space and to an observer moving uniformly with the vortices. The results are presented in five diagrams.

C. C. Lin (Pasadena, Calif.).

Serebrijsky, J. M. Flow past bodies of revolution. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 99–108 (1944). (Russian. English summary) [MF 11600]

An approximate method is considered in this work which permits the construction of the distribution of pressures of an incompressible liquid on the surface of a body of revolution. The same method may be used to construct a field of velocities for this flow.

Author's summary.

Lambin, N. V. Discontinuous flow past a lattice of broken profiles. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 187–200 (1944). (Russian. English summary) [MF 11594]

The paper is concerned with the problem of determining pressure and velocity distributions which the steady flow of an ideal incompressible fluid exerts on a lattice of broken rigid profiles.

Author's summary.

Gareaud, Louis. Sur la résistance opposée par l'air à une surface ogivale en régime permanent. C. R. Acad. Sci. Paris **217**, 259–261 (1943). [MF 11111]

The author discusses the distribution of pressure and velocity on the ogival head of a projectile in flight. He derives approximate formulas, one set for subsonic, one for "trans-sonic" and one for supersonic velocities. These formulas are obtained by intuitive arguments from the known theories of von Kármán, Meyer and Taylor.

C. B. Morrey, Jr. (Aberdeen, Md.).

Isaacs, Rufus. Airfoil theory for flows of variable velocity. J. Aeronaut. Sci. **12**, 113–117 (1945). [MF 11592]

For the two-dimensional case of a wing moving with variable speed through a stationary incompressible fluid, the author deduces the integral equation that relates the circulation with the velocity. [Compare Wagner [Z. Angew. Math. Mech. **5**, 17–35 (1925)] or Sears [J. Franklin Inst. **230**, 95–111 (1940); these Rev. **2**, 28].] In the case of the rotor of a rotary-wing aircraft (to which the two-dimensional theory is approximately applicable), the forward speed is the sum of a constant and a sinusoidal term. The author works out this case in detail and obtains a formula for the lift, involving a rather complicated infinite series of Bessel functions. He also presents a numerical example typical of current helicopter practice, which shows that the nonstationary effect on the lift is small.

W. R. Sears.

Steinhardt, F. Note on the elliptic wing. Quart. Appl. Math. **2**, 346–347 (1945). [MF 11776]

Goodey, W. J. Two-spar wing stress analysis. The mathematical theory, with numerical examples. Aircraft Engrg. **15**, 2–7, 38–41, 46 (1943). [MF 8917]

This paper deals with a two-spar, skin-covered wing with linear tapering. The skin is regarded fundamentally as a shear carrying member. A bending moment m is added to the front spar and subtracted from the rear spar by the action of the skin covering. The strain energy U of the wing is computed, taking into account only the shear in the skin covering and the bending moment in the spars; U is minimized by use of the calculus of variations, and a differential equation satisfied by m is obtained. This equation is linear and of the second order. Its solution is reduced

to quadratures, and the integrations involved are carried out numerically. Wings are considered consisting of various combinations of covered and open sections. The effect of the skin covering on the leading edge is first neglected, and later taken into account. Finally, the effect of the tensile stress in the skin covering on the underside of the wing is taken into account. *G. E. Hay* (Providence, R. I.).

Brodetsky, S. The general motion of the aeroplane. Philos. Trans. Roy. Soc. London. Ser. A. **238**, 305–355 (1940). [MF 11506]

The author attempts a systematic study of the equations of motion of an airplane. The mathematical technique is the same as that used in an earlier paper presented before the 5th International Congress for Applied Mechanics at Cambridge, Mass., 1938 [Proceedings, pp. 599–605]. The scope of the present investigation is best characterized by the following chapter and section headings. (I) Longitudinal Motion without Screw Thrust. Equations of motion, coefficients of statical and dynamical stability κ, τ; first approximation. The three standard conditions of the symmetrical aeroplane. Longitudinal stability; usual value of τ in standard normal condition. Standard normal condition: κ of zero order, Lanchester's phugoids; κ small, extended phugoids; κ negligible, neutral phugoids; second approximation, Lanchester's phugoids, the loop. Standard diving condition, diving phugoids. Elevator in rotation during motion, flattening out from a dive. Lanchester's phugoids corrected for drag. (II) Longitudinal Motion with Engines in Action. Equations of motion, first approximation. Standard normal condition: moderate power, Lanchester's, extended and neutral phugoids, large power, power phugoids, zooming. (III) Three-dimensional Motion. Symmetrical aeroplane: equations of motion, first approximation. Standard normal condition; κ of zero order, three-dimensional phugoids, Immelmann turn; κ small, extended three-dimensional phugoids; κ negligible. Standard stalled conditions, the slow spin. Aeroplane with displaced controls, additional moments. Standard normal condition: κ of zero order; small asymmetry, three-dimensional phugoids; large aileron displacement, the slow roll. *W. Prager* (Providence, R. I.).

Goldstein, S. On the limiting values for infinite pitch of a parameter occurring in airscrew theory. Proc. Cambridge Philos. Soc. **40**, 146–150 (1944). [MF 10784]

The parameter mentioned in the title of the paper is given by the formula

$$K=K(x)=\frac{N\tau}{\pi^2x^2}P\int_0^\infty \frac{dt}{(t^2-\tau^2)(1+t^2)^n},$$

where $N>2$, $\tau=x^{-\frac{1}{2}N}(1-x^N)^{\frac{1}{2}}$, $n=2/N$ and P denotes the principal value of the integral. By using complex integration $K(x)$ is expressed in terms of hypergeometric functions. In applications N is an integer, denoting the number of blades of an airscrew. The final formulae include $N=2$ as a limiting case. *A. Weinstein* (Toronto, Ont.).

Theory of Elasticity

Thomas, T. Y. Remark on a distortion tensor for elastic displacements. Proc. Nat. Acad. Sci. U. S. A. **30**, 140–143 (1944). [MF 10742]

One of the useful conditions for yield, due to von Mises, states that yield will occur when the energy per unit volume produced by a change of shape alone reaches a certain value characteristic of the material under consideration. This leads the author to seek a distortion tensor which will express the condition of yield directly in terms of this tensor. He defines a tensor D, which he calls the distortion tensor, as an invariant of an arbitrary elastic displacement; and $D=0$ if, and only if, the displacement is conformal. The tensor E is defined as the distortion analogue of the stress tensor and, except for a constant factor, is the energy per unit volume due to distortion. Since the squares of the tensors D and E are shown to differ only by a constant factor, the von Mises yield condition is restated as follows: yield will occur when the square of the distortion tensor D (or its stress analogue E) attains a characteristic value. The author indicates that various other useful forms of the yield conditions can be obtained. *A. Gelbart* (Syracuse, N. Y.).

Thomas, T. Y. Surfaces of maximum shearing stress. J. Math. Phys. Mass. Inst. Tech. **23**, 167–172 (1944). [MF 11410]

It is well known that at each point of an elastic medium in a state of equilibrium there are at least two local plane elements having maximum shearing stress. In this paper the author is primarily concerned with the case where there are exactly two such plane elements. His problem is to determine the conditions satisfied by the stress tensor $(\sigma_{\alpha\beta})$ when a set of these plane elements can be united so as to generate a family of surfaces. The method used consists in: (1) determining a formula for the covariant derivatives of the unit vectors (ν^α) in the directions of principal stress; (2) expressing the unit vector field (v^α) perpendicular to the above local plane elements in terms of the unit vectors (ν^α); (3) using (2) in the formula $\epsilon^{\alpha\beta\gamma}v_\alpha v_{\beta,\gamma}=0$, which expresses the condition that the (v^α) generate families of surfaces. It is shown that the conditions on the covariant derivatives of the stress tensor $(\sigma_{\alpha\beta,\gamma})$ and principal stresses (τ_i) are (for one family)

$$\frac{\sigma_{12,3}-\sigma_{12,1}}{\tau_1-\tau_2}-\frac{2\sigma_{13,2}}{\tau_1-\tau_3}+\frac{\sigma_{23,1}-\sigma_{32,3}}{\tau_2-\tau_3}=0$$

and (for the other family)

$$\frac{\sigma_{12,3}+\sigma_{12,1}}{\tau_1-\tau_2}-\frac{2\sigma_{13,2}}{\tau_1-\tau_3}+\frac{\sigma_{23,1}+\sigma_{32,3}}{\tau_2-\tau_3}=0.$$

N. Coburn (Austin, Tex.).

Allen, D. N. de G. and Southwell, R. V. Relaxation methods applied to engineering problems. X. The graphical representation of stress. Proc. Roy. Soc. London. Ser. A. **183**, 125–134 (1944). [MF 11499]

Süray, Saffet. Sur les lignes de tension principale. Rev. Fac. Sci. Istanbul. (A) **6**, 40–43 (1941). (French. Turkish summary) [MF 10814]

The author starts from W. Prager's representation of any two-dimensional system of stresses in a homogeneous isotropic elastic disc by the aid of two functions of a complex variable [Rev. Math. Union Interbalkan. **3**, 63–65 (1941); these Rev. **3**, 29] and proves the following theorem. Consider two families of curves, which intersect each other at an angle ϑ, which satisfies the Laplace equation. If one of the systems of curves constitutes a set of principal stress trajectories of an elastic system, then the other constitutes a set of principal trajectories of another elastic system of the same type. Furthermore, the author states the following

theorem. Linear conformal transformations are the only conformal transformations which map any net of stress trajectories of a plane homogenous elastic disc onto another net of the same kind. *P. Neményi* (Pullman, Wash.).

Akopian, A. A. Applications of thermodynamics to the equilibrium of ideal elastic systems. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 225–240 (1944). (Russian. English summary) [MF 11596]

Beskin, Leon. General solution of two-dimensional problems of elasticity. J. Appl. Phys. **15**, 562–567 (1944).

This paper deals with a two-dimensional simply connected domain with an arbitrary boundary and given body and surface forces. By use of the superposition principle, the problem is reduced to one with vanishing body forces and given surface forces. For this latter problem the determination of the Airy stress function (which is a biharmonic function) is reduced by the use of certain aspects of the theory of potential to the solution of several integral equations.

G. E. Hay (Providence, R. I.).

Sherman, D. I. Plane deformation in isotropic inhomogeneous media. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 301–309 (1943). (Russian. English summary) [MF 11231]

Let S and Q be simply connected regions bounded by curves L and γ, respectively, and let Q be inside S. Let Q be filled with one medium, $S-Q$ with another possessing different elastic constants. The author gives a new solution for the determination of the elastic state in Q from given exterior forces on L under the assumption that the elongations are prescribed at every point of γ. He reduces the problem to a system of linear integral equations of Fredholm type. The problem was solved previously by a more complicated method by Michlin and by the author [Trudy (Works) Seismol. Inst. Acad. Sci. USSR **1935**, no. 66 and **1938**, no. 86, respectively]. *S. Bergman.*

Girkmann, K. Zum Halbebenenproblem von Michell. Ing.-Arch. **14**, 106–112 (1943). [MF 11433]

This is a detailed discussion of the so-called Michell problem [Proc. London Math. Soc. **32**, 35–61 (1900)] of the two-dimensional stress distribution due to a concentrated load acting normal to the straight boundary of a half-plane. It is shown that the results can also be derived from the well-known solution of Boussinesq for the analogous three-dimensional problem by superposition. The aim of this paper is to dispel the doubt expressed by J. Ohde [Bauingenieur **20**, 451 (1939)] on the validity of Michell's solution for all values of Poisson's ratio. *H. S. Tsien.*

Hruban, K. Der Spannungszustand des im Innern beanspruchten Halbraumes. Ing.-Arch. **14**, 9–13 (1943). [MF 11427]

The author gives the following solutions for the isotropic elastic half-space with a plane boundary: (1) radial shearing force distribution over the boundary; (2) pressure load acting on a spherical surface completely contained in the half-space; (3) concentrated force acting at a point in the half-space. These solutions are said to be useful in soil mechanics and the theory of concrete structures. However, it seems that the disturbance in the pressure load on the spherical surface for (2) due to the stress function introduced to satisfy the conditions at the plane boundary is not considered. *H. S. Tsien* (Pasadena, Calif.).

Borowicka, H. Die Druckausbreitung im Halbraum bei linear zunehmendem Elastizitätsmodul. Ing.-Arch. **14**, 75–82 (1943). [MF 11431]

The problem is that of a concentrated load acting normal to the plane boundary of a stratified isotropic elastic half-space simulating the earth. If x is the distance from the plane boundary, then the shear modulus G is assumed to be equal to Cx, where C is a constant. Poisson's ratio is $1/m$, with m constant. The general solution is obtained as an infinite series. It is shown that the stresses in the half-space are independent of C. For $m=2, 3$ the infinite series are reduced to single terms. For $m=2$, the stresses are the same as those of the usual case with $G=$constant. For other values of m the stresses are different. For example, the direct stress σ_x in the x-direction tends to concentrate at the region underneath the load with increasing m, while, for the case of $G=$constant, σ_x is independent of m.

H. S. Tsien (Pasadena, Calif.).

Borowicka, H. Über ausmittig belastete, starre Platten auf elastisch-isotropem Untergrund. Ing.-Arch. **14**, 1–8 (1943). [MF 11426]

The author investigates the problem of an infinitely long strip of rigid plate and of a rigid circular plate resting on the surface of isotropic elastic ground and loaded off-center by a concentrated force. As is usual in treating such problems, the frictional force between the plate and the ground is neglected. The reaction pressure from the ground is determined by first assuming expressions for the pressure distributions and then determining the unknown coefficients in such expressions by the conditions imposed on the displacements of the plates by their rigidity. It is shown that if a is the half-width of the strip or the radius of the circular plate and e is the distance between the load and the center of the plates, then the reaction at the farther corner of the plate changes from compression to tension at $e/a=\frac{1}{2}$ for the strip and at $e/a=\frac{1}{3}$ for the circular plate.

H. S. Tsien (Pasadena, Calif.).

Schapiro, G. S. Distribution of stresses in an infinite layer. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 167–168 (1944). (Russian. English summary) [MF 11603]

The author considers the problem of an elastic layer resting on a smooth incompressible surface and subjected to the action of a normal load p uniformly distributed over a circle. The solution is obtained by means of a stress function which follows from the Galerkin solution for a thick circular plate. The final formulae for stresses and displacements are expressed by means of Fourier integrals. The numerical results obtained by means of approximate computations are given graphically. For ρ approaching zero the results of Marguerre follow as a particular case. *Author's summary.*

Beskin, Leon. Theory of membranes in shape of developable surfaces. J. Appl. Phys. **15**, 547–561 (1944).

Use is made of the fact that a developable surface can be generated by the tangents to a skew curve (the edge of regression of the surface). Orthogonal surface coordinates on the surface are introduced, and the three equations of equilibrium are written in terms of the three components of the stress flow (stress multiplied by thickness). These equations are integrated when the surface loading varies as a power p of the distance from the edge of regression. The solution contains two arbitrary functions which are determined from the edge conditions. The effect of various values of p is discussed. The case of vanishing surface loading is

considered in some detail. Practical application of the theory is made to stressed-skin structures. *G. E. Hay.*

Wansleben, F. Die Beulfestigkeit rechteckig begrenzter Schalen. Ing.-Arch. **14**, 96–105 (1943). [MF 11432]

The problem is that of a shell of double curvature with a simply supported rectangular boundary. The principal curvatures, in directions parallel to the sides of the rectangle, are assumed to be constant and small compared with the dimensions of the shell. The thickness is constant. The loads are constant compressions applied at the boundary. The buckling calculated with infinitesimal displacements is easily determined under such assumptions and a general criterion is obtained. It is shown that the general result reduces to that of the known investigations by von Mises for the laterally loaded cylindrical shell, by von Sanden, Tölke, Flügge, etc., for the axially loaded cylindrical shell and by Zoelly and Schwerin for the spherical shell. However it must be kept in mind that in many cases the buckling loads calculated by assuming infinitesimal displacements are the "upper buckling loads" which are much higher than the practically important "lower buckling loads" [J. Aeronaut. Sci. **9**, 373–384 (1942)]. *H. S. Tsien* (Pasadena, Calif.).

Sen, Bibhutibhushan. Boundary value problems of circular disks under body forces. I. Bull. Calcutta Math. Soc. **36**, 58–62 (1944). [MF 11351]

Sen, Bibhutibhusan. Boundary value problems of circular disks under body forces. II. Bull. Calcutta Math. Soc. **36**, 83–86 (1944). [MF 11845]

The author proposes to determine the stresses in a heavy circular disk which rotates with constant speed about a horizontal axis. The problem actually solved, however, is a slightly different one, namely, the determination of the stresses in a stationary circular disk under the influence of gravitational and centrifugal body forces. [The reviewer doubts that the original dynamical problem can be treated satisfactorily in this statical manner. As the disk rotates its elements change their position with respect to the stationary field of gravitational and centrifugal forces. This change is necessarily accompanied by vibrations.] *W. Prager.*

Karcivadze, I. Fundamental problems of the theory of elasticity for an elastic circle. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **12**, 95–104 (1943). (Georgian. Russian summary) [MF 11687]

Federhofer, K. und Egger, H. Knickung der auf Scherung beanspruchten Kreisringplatte mit veränderlicher Dicke. Ing.-Arch. **14**, 155–166 (1943). [MF 11435]

The problem of the buckling of an annular plate of constant thickness under uniformly distributed tangential shearing stress has been treated by W. R. Dean [Proc. Roy. Soc. London. Ser. A. **106**, 268–284 (1924)]. In the present paper, the thickness of the plate is taken to vary radially according to a power of the distance from the center. The situation is in this case essentially more complicated than in the former because the solution of the differential equation of buckling can only be expressed in infinite series while for the plate of uniform thickness this solution can be expressed in closed form. The consequence is that the elements of the determinant, whose vanishing determines the buckling stress, are infinite series. A table and families of curves give the critical buckling stress as a function of the ratio of the inner to the outer radius of the annulus for the variation of the thickness according to three different powers of the radius. *H. W. March* (Madison, Wis.).

Doucet, E. Remarques sur la solution donnée par Navier au problème des plaques rectangulaires. Génie Civil **119**, 316–317 (1942). [MF 10937]

The problem in question concerns a rectangular plate simply supported along the four edges and acted upon by an unspecified normal load. Remarks are presented to support a claim that the classical solution to this problem is incorrect. These remarks appear to be based on the incorrect assumption that a function of two variables cannot be developed in a rectangle in a double Fourier series containing only sine functions. *G. E. Hay* (Providence, R. I.).

Mikeladze, Š. E. New finite difference equations for the computation of rectangular plates, freely supported along the boundaries. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **4**, 737–744 (1943). (Georgian. Russian summary) [MF 11708]

Hogg, A. H. A. Equilibrium of a thin slab on an elastic foundation of finite depth. Philos. Mag. (7) **35**, 265–276 (1944). [MF 10859]

The system considered is composed of a uniform elastic slab, which covers the whole of a perfectly rough, rigid plane, and a thin elastic plate of infinite extent which rests on the elastic slab and is subjected to a single concentrated force normal to the plate. It is assumed that the slab and the rigid plane, as well as the slab and the plate, remain in contact with each other when the load is applied. The displacements and stresses are given numerically for certain ranges of the parameters. It is indicated that the theory may be useful in interpreting the results of measurements made on concrete slabs used for foundations.

J. J. Stoker (New York, N. Y.).

Kosko, Eryk. On the treatment of discontinuities in beam deflection problems. Quart. Appl. Math. **2**, 271–272 (1944). [MF 11136]

Historical remarks in connection with a paper by C. L. Brown [Quart. Appl. Math. **1**, 349–351 (1944); these Rev. **5**, 251] on the same subject. *E. Reissner.*

Dörr, J. Der unendliche, federnd gebettete Balken unter dem Einfluss einer gleichförmig bewegten Last. Ing.-Arch. **14**, 167–192 (1943). [MF 11436]

The deflections of an elastically supported semi-infinite beam under the influence of a uniformly moving point load are investigated. The problem is treated as a linear one, by methods depending on the Laplace transform and the Fourier integral. When the velocity of the load is less than a certain critical value, the solution is comparatively simple, but when it exceeds this critical value the situation is more complicated, and the solution can only be reduced to power series in $\sqrt{t}$. *P. Franklin* (Cambridge, Mass.).

Mandel, Jean. Sur l'instabilité d'une tige élastique par torsion ou par flexion. C. R. Acad. Sci. Paris **217**, 667–668 (1943). [MF 11679]

Vazsonyi, Andrew. A numerical method in the theory of vibrating bodies. J. Appl. Phys. **15**, 598–606 (1944).

Numerical methods for the determination of the normal modes and frequencies of linear elastic systems are illustrated by a detailed treatment of a specific problem. The problem is that of a stretched membrane in the shape of a trapezoid. The differential equation for the deflection w is replaced by appropriate finite difference equations, which

are to be satisfied at the points of a net covering the trapezoid. Arbitrary values are assumed for w at the interior net points, which are then corrected step-wise by using relaxation methods in such a way as to obtain the eigenvalue for a given mode of vibration as well as the deflection. The methods chosen to do this appear to converge rapidly for the second mode and frequency as well as the fundamental. A similar method for treating forced vibrations is indicated.
J. J. Stoker (New York, N. Y.).

Lehr, Georges. Sur les fréquences propres des arbres vibrant en torsion. C. R. Acad. Sci. Paris **217**, 421–422 (1943). [MF 11662]

This is a sequel to a previous note by the same author [C. R. Acad. Sci. Paris **217**, 285–287 (1943); these Rev. **6**, 84]. A further study is made of the algebraic equation which determines the frequencies of the torsional vibrations of a system consisting of n masses mounted on a shaft. It is shown that the equation can be formed by means of a system of recurrence relations (instead of by expanding a determinant).
L. A. MacColl (New York, N. Y.).

Boukidis, N. A. and Ruggiero, R. J. An iterative method for determining dynamic deflections and frequencies. J. Aeronaut. Sci. **11**, 319–328 (1944). [MF 11253]

An iteration method based upon a procedure due to Biezeno-Grammel for obtaining the eigenvalues of an elastic beam is presented. Let $\psi_0(x)$ be an arbitrary deflection curve of the beam satisfying the boundary conditions. Consider now the system statically loaded by a force $p_1(x)$ given by $p_1(x)=m(x)\psi_0(x)$, where $m(x)$ is the mass distribution. This load produces a deflection $\psi_{11}(x)$. Continuing the process, put $p_k(x)=m(x)\psi_{1,k-1}(x)$. The first (lowest) eigenvalue ω_1 is then obtained from

$$\omega_1^2=\lim_{k\to\infty}(\psi_{1,k}(x)/\psi_{1,k+1}(x)).$$

An application to the oscillation of an airplane wing and a scheme for carrying out practical computations are given.
H. W. Liepmann (Pasadena, Calif.).

Ishlinsky, A. J. Stability of plastico-viscous flow of a circular plate. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 405–412 (1943). (Russian. English summary) [MF 11260]

The author studies the flow in a circular plastic disk of variable thickness exposed to normal forces which are uniformly distributed over the circular contour. The method is that developed in a previous paper by the same author [Appl. Math. Mech. **7**, 109–130 (1943); these Rev. **5**, 252].
W. Prager (Providence, R. I.).

Coburn, N. A boundary value problem in plane plasticity for the Coulomb yield condition. J. Math. Phys. Mass. Inst. Tech. **23**, 117–125 (1944). [MF 11138]

This paper deals with the problem of plane stress for the half plane, with yield condition $\sigma_1-\sigma_2=(\sigma_1+\sigma_2)b+2k$. The author considers the system of successive differential equations which is obtained if a solution as power series in b is attempted.
E. Reissner (Cambridge, Mass.).

Illiushin, A. A. Approximate theory of the elastico-plastic deformation of shells with the axial symmetry. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 15–24 (1944). (Russian. English summary) [MF 11466]

reviewed on p. 252
The method developed in the paper ~~reviewed above~~ is applied to axially symmetric deformations of thin shells of revolution.
W. Prager (Providence, R. I.).

Carter, G. K. Numerical and network-analyzer solution of the equivalent circuits for the elastic field. J. Appl. Mech. **11**, A-162–A-167 (1944). [MF 11018]

This paper indicates some numerical applications of the paper reviewed below.
A. E. Heins.

Kron, Gabriel. Equivalent circuits of the elastic field. J. Appl. Mech. **11**, A-149–A-161 (1944). [MF 11017]

The author shows how one may replace the study of the equations of elasticity by the study of the equations of lumped circuit phenomena. The elastic body may be nonhomogeneous, may be subject to arbitrary boundary conditions or may have fairly arbitrary shape. With this particular technique, such problems as the propagation of elastic waves, the determination of stresses and strains in a steady stressed body, etc. may be solved to a reasonable degree of accuracy. A similar technique has been applied by the author to related problems in electromagnetic theory [G. Kron, Proc. I. R. E. **32**, 289–299 (1944); these Rev. **6**, 55].
A. E. Heins (Cambridge, Mass.).

Kron, Gabriel. Tensorial analysis and equivalent circuits of elastic structures. J. Franklin Inst. **238**, 399–442 (1944). [MF 11487]

Equivalent circuits are developed to represent one, two or three dimensional elastic structures. The elastic structures are assumed to be rigid bodies having different elastic coefficients along three perpendicular axes, and long thin beams. These beams and bodies are connected rigidly or by various constraints into a mechanical structure. For small steady deformations and forced or natural vibrations the circuit equations may be solved by the A. C. network analyzer. Southwell's relaxation method can also be used to obtain an approximate numerical solution [cf. the preceding review].
A. E. Heins (Cambridge, Mass.).

Carter, G. K. and Kron, Gabriel. Network analyzer solution of the equivalent circuits for elastic structures. J. Franklin Inst. **238**, 443–452 (1944). [MF 11488]

The equivalent circuits developed for elastic structures in the paper reviewed above are checked for two known structures by the A. C. network analyzer. In each case, the measured values check satisfactorily with the calculated values.
A. E. Heins (Cambridge, Mass.).

BIBLIOGRAPHICAL NOTE

Universidad Nacional de la Plata. Publicaciones de la Facultad de Ciencias Fisicomatematicas. Serie Segunda. Revista y Contribuciones.

Since the explanation of the system of numbering peculiar to this journal in these Rev. **6**, 56, the following change has been introduced. The "Serie Segunda" is no longer synonymous with "Revista" but contains two independent series, "Revista" and "Contribuciones"; the former is to appear bimonthly, the latter irregularly.

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AUTHOR INDEX

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